FINAL CHECKLIST

1. Topics Outline

Listed below is an outline of topics that we have covered in this course. Numerical methods and the derivation of heat/wave/Laplace/Sturm-Liouville equations are not in the list, even though they are also covered.

1.1. First Order ODEs.

1.**Linear Equations**: Integrating factors. Idea: For the equation $y' + p(x)y + q(x) = 0$, find a function $\mu(x)$, such that whenever $y(x)$ is a solution to the equation, $z(x) = \mu(x)y(x)$ solves an equation of the form $z' + \tilde{q}(x) = 0$.

2.**Separable Equations**: Equations which can be put in the form $p(x) + q(y)\frac{dy}{dx} = 0$. Often, we obtain implicit solutions for separable equations.

3.**Solving non-separable equations via separable ones**: e.g. consider the equation $\frac{dy}{dx} = \frac{x^2+xy+y^2}{xy}$, and use the substitution $z = \frac{y}{x}$.

4.**Autonomous Equations**: Vector field plot and phase-line analysis. Equilibrium and stability.

5.**Existence & Uniqueness Theorem**: Statement of the Theorem. Note: theorem is about initial value problems; the conclusion is local, nearby our initial value.

6.**Modeling**: Keys are, identify the unknown function, and use the relations that can be derived from the statement of the problem to establish equations.

7.**Exact Equations**: Testing exactness. If exact, how to find solutions? If not exact, is there any test that can tell us whether we can multiply the entire equation by some function and make the equation exact?

1.2. Second Order Linear ODEs.

1.**Constant Coeff., Homogeneous**: Characteristic Polynomial.

2.**Constant Coeff., Non-homogeneous**: Undetermined Coefficients; or Variation of Parameters.

3.**Non-constant Coeff., Homogeneous**: One non-zero solution known, find another solution (linearly independent from the one already known) by Reduction of Order.

4.**Non-constant Coeff., Non-homogeneous**: From knowing a fundamental set of solutions of the underlying homogeneous equation to a particular solution of the non-homogeneous equation—*Variation of Parameters.*

1.3. Series Solutions of ODEs (at Ordinary Points).

1.**Power Series**: Definition of Power series about the point $x = x_0$; Radius of Convergence; Ratio test; Taylor expansion; Analytic Functions.

2.**Method of Undetermined Coefficients**: Expand the solution $y(x)$ about the point $x = x_0$ where the initial conditions are set, then use the equation to determine the coefficients in the expansion.

3.**Shift of Index and Recurrence Relation**: The summation notation (and shift of index) is useful only when we are interested in knowing more than a few terms in the series solution.

1.4. **Euler Equations.** Identify the form of an Euler equation; Three cases of solutions.

1.5. **Laplace Transform.**

1.**Definition and Calculation**: The Laplace transform; Laplace transform of elementary functions; First and second shifting theorems; Derivative properties; Inverse Laplace transform.

2.**Step Functions and Impulse Functions**: Definitions of $u_c(t)$ and $\delta(t)$; their Laplace transform.

3.**Convolution Integral**: Definition; Crucial theorem: $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

4.**Laplace Transform and ODEs**: We only considered the case when the ODE has constant coefficients, but the forcing function can be various kinds.

1.6. **Two-point Boundary Values problems.**

The concept of *eigenvalues* and *eigenfunctions*.
Solving Principle: Find the general solutions of the equation; then use the boundary values to determine whether the solutions can be nonzero.

1.7. **Fourier Series.**

1.**Definition and Calculation**: Periodic functions; Definition of Fourier series; Formulas for calculating the Fourier coefficients.

2.**Fourier Convergence Theorem**: Statement of the theorem; Given a periodic function (maybe discontinuous), which function does its Fourier series converge to?

3.**Sine and Cosine Series**: Even and odd functions and properties (especially the integral over the interval $[-L, L]$); Expanding $f(x)$ $(0 \leq x < L)$ as a cosine or sine series of period $2L$.

1.8. **Heat, Wave, Laplace Equations.**

1.**Definitions**: Heat equation (with ends with fixed constant temperature or insulated ends); The meaning of "steady state"; Wave equation (with zero initial velocity or position, but you need to know how to bring them together); Laplace equation (in a rectangle or a round disk);

2.**Solution Technique**: The principle of solutions for these equations is three steps: Separation of variables, obtaining two ODEs; Consider the homogeneous boundary/initial values, giving us a two-point boundary value problem for one of the ODEs; use the remaining boundary/initial conditions, together with the principle of superposition to find the formal solution.

1.9. **Sturm-Liouville BVP.**

1.**Definitions**: The form of a Sturm-Liouville problem (including the linear operator L, which turns (SL) into the form $L[y] = \lambda r(x)y$); Hermitian L^2-inner product $\langle u, v\rangle_H$; L^2-inner product $\langle u, v\rangle_{r(x)}$; orthogonality of functions under $\langle \cdot, \cdot\rangle_{r(x)}$; normalized eigenfunctions;

2.**Theorems**: Lagrange identity (i.e., $\langle L[u], v\rangle_H = \langle u, L[v]\rangle_H$, where u, v satisfy the BV of (SL)) and its proof; Using the Lagrange identity to show all eigenvalues of (SL) are real, and that eigenfunctions corresponding to different eigenvalues are orthogonal under $\langle \cdot, \cdot\rangle_{r(x)}$; The statement and meaning of Theorems 4-5.

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2. Sample Review Questions

Here are some sample review questions for your reference, which are barely organized and, of course, not complete.

1. What are the types of first order ODEs you know how to solve? How about second order ODEs?

2. We have seen two versions of *integrating factors*, what are they used for respectively and, in principle, how to find them?

3. What's common in the ideas of reduction of order and variation of parameters?

4. Can you reproduce the entire list of Laplace transforms in the book using as few known formulas/properties as possible?

5. What is the idea of using Laplace transform to solve constant coefficient ODEs?

6. How do you understand the impulse function? How does it relate to other functions, such as the step function?

7. When do we need to use the summation notation in the using the series method? When is it easier not to use it?

8. For what kind of equations (initial-boundary value problems) can we superpose its solutions and still get solutions of the same equation?

9. In the wave equations, why do we say, in order to solve the equation with general initial values, it is sufficient to know solutions of the following two cases: zero initial velocity, zero initial position?

10. Given a function $f(x)$, periodic with period $2L$, and f, f' both piecewise continuous, which is the limit of its Fourier series expansion?

11. Can we say that the Theorem 5 (convergence theorem of the ψ_n-series, where ψ_n are all the eigenfunctions of (SL)) for the Sturm-Liouville problem is a generalization of the

Fourier convergence theorem? Why?

12. Briefly explain why in the heat equation in a rod, the boundary condition $u(0,t) = u(L,t) = 0$ will give us $X_n(x)$ as sine functions, while $u_x(0,t) = u_x(L,t) = 0$ will give us $X_n(x)$ as cosine functions.

13. Briefly explain why in the wave equation, the zero initial velocity gives us $T_n(t)$ as cosine functions, while the zero initial position gives us $T_n(t)$ as sine functions.

14. In Sturm-Liouville problems, what does it mean to say the eigenfunctions are *normalized*? What is the underlying inner product? How to normalize an eigenfunction? Can you derive this formula by definition?

15. In simple cases such as $p(x) = r(x) = 1$, with boundary values $y(0) = y(1) = 0$, can you show that the linear operator L defined in (SL) is self-adjoint?