FINAL EXAM, MATH 353 SUMMER I 2015

9:00am-12:00pm, Thursday, June 25

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: This exam contains **12 pages** and **10 problems** with a table of Laplace transform at the end. You have **180 minutes** to answer all the questions. You may use a calculator or a review sheet, front and back, written in your own handwriting. Throughout the exam, show your work with clear reasoning and calculation. If you are using a theorem to draw some conclusions, quote the result. If you do not completely solve a problem, explain what you understand about it. No collaboration on this exam is allowed. *Good luck* !

Problems	Points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20 points) Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the following functions:

(1) $F(s) = \dfrac{s-2}{s^2+2s+10}$

(2) $F(s) = \dfrac{e^{-s}}{s^3}$

(3) $F(s) = \dfrac{1}{(s^2-4)(s^2-9)}$

2. (20 points) Find the solution of the initial value problem

$$y'' + 4y = 8\sinh(2t) + \delta(t - 2\pi) \qquad y(0) = y'(0) = 0,$$

where $\delta(t)$ is the Dirac delta function and $\sinh(t)$ is the hyperbolic sine function.

3. (20 points) Consider the first order ordinary differential equation

$$e^x(x+1) + (ye^y - xe^x)\frac{dy}{dx} = 0.$$

(1) Find an integrating factor $g(y)$ which makes this equation *exact*.

(2) Find the (implicit) solution of the differential equation above with $y(0) = 1$.

4. (20 points) Find the general solution of the equation

$$(x+1)y'' - xy' - y = 0, \qquad x > -1$$

given that $y(x) = e^x$ is a solution. Your final answer could involve an indefinite integral.

5. (20 points) Use the series method to solve the initial value problem

$$y'' + 3xy' + e^x y = 2x, \qquad y(0) = 1,\ y'(0) = -1.$$

You need only to calculate terms up to the *fourth* power of x.
Note: The Taylor expansion of e^x at $x = 0$ is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$

6. (20 points) Show by calculation that the Fourier series for the function

$$f(x) = e^x, \qquad -\pi \le x < \pi,\ f(x) = f(x + 2\pi)$$

is

$$g(x) = \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2}(\cos nx - n \sin nx)\right).$$

What is the value of $g(\pi)$? Now find the limit of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2}.$$

7. (20 points) Consider the initial-boundary value problem:

$$\begin{cases} u_{xx} = u_t - \sin x, & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 3\sin 2x + 6\sin 5x, & 0 \le x \le \pi \end{cases}$$

(1) Find the **steady state** solution of this equation, i.e., a solution $w(x,t) = w(x)$ (does not depend on t) which satisfies the equation and the boundary values.

(2) Find the solution $u(x,t)$ of the initial-boundary value problem.
Hint: Which initial-boundary value problem does $v(x,t) = u(x,t) - w(x)$ satisfy?

8. (20 points) Find the solution $u(x,t)$ of the following wave equation problem:

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin 3x, \ u_t(x,0) = \sin 4x, & 0 \le x \le \pi \end{cases}$$

9. (20 points) Find the solution $u(r,\theta)$ of the following Laplace equation in the *semicircular* region $r < 1,\ 0 < \theta < \pi$:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 \le r < 1,\ 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0, & 0 \le r < 1 \\ u(1,\theta) = f(\theta), & 0 \le \theta \le \pi, \end{cases}$$

assuming that $u(r,\theta)$ is single-valued and bounded in the given region.
Note: You are allowed to quote intermediate results in our solution of a similar problem with the region being *circular*.

10. (20 points) Consider the boundary value problem

$$\begin{cases} [(1+x^2)y']' + y = \lambda(1+x^2)y \\ y(0) - y'(1) = 0, \quad y'(0) + 2y(1) = 0. \end{cases}$$

Let L be defined by $L[y] = [(1+x^2)y']' + y$. Prove that for any $u(x), v(x)$ satisfying the *boundary conditions* above, we have

$$\langle L[u], v\rangle_H = \langle u, L[v]\rangle_H,$$

where $\langle u, v\rangle_H$ denotes the *Hermitian* L^2*-inner product* of u, v on the interval $[0, 1]$.
Hint: You may use integration by parts *twice*.

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \qquad s > 0$
e^{at}	$\frac{1}{s-a}, \qquad s > a$
t^n ($n > 0$, integer)	$\frac{n!}{s^{n+1}}, \qquad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \qquad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \qquad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \qquad s > 0$
$\cosh at$	$\frac{s}{s^2-a^2}, \qquad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \qquad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \qquad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \qquad s > 0$
$e^{ct}f(t)$	$F(s-c)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \qquad c > 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$