EXAM I, MATH 353 SUMMER I 2015

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test contains **9 pages** and **6 questions**. You have **75 minutes** to answer all the questions.
Good Luck !

Useful Formulas:

$$y_p(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(s)g(s)}{W(y_1,y_2)}ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)g(s)}{W(y_1,y_2)}ds,$$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

You are responsible for identifying correctly the situation in which these formulas can be applied.

Question	Max. Points	Score
1	25	
2	15	
3	20	
4	20	
5	15	
6	5	
Total	100	

1. (25 points) Consider the differential equation

$$y^2 + y - x\frac{dy}{dx} = 0.$$

(1) Show that this equation is *not exact.*

Solution. The equation is already in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

We have

$$\frac{\partial M}{\partial y} = 2y + 1, \qquad \frac{\partial N}{\partial x} = -1,$$

which are not equal.
Therefore, the above equation is not exact.

(2) Find an integrating factor for this equation.
Hint: There exists an integrating factor $v(y)$. Also, before proceeding to the next question, you may want to check that the original equation becomes exact when multiplied by the integrating factor you obtained.

Solution. Suppose that the equation becomes exact after being multiplied by a function $v(y)$. The equation now looks like:

$$v(y)(y^2 + y) - xv(y)\frac{dy}{dx} = 0.$$

For this to be exact, we need

$$\frac{\partial}{\partial y}[v(y)(y^2 + y)] = \frac{\partial}{\partial x}(-xv(y)).$$

This is equivalent to

$$(y^2 + y)v' + (2y + 1)v = -v,$$

and simplifies to

$$\frac{v'}{v} = -\frac{2}{y}.$$

Hence, a choice of $v(y)$ is

$$v(y) = y^{-2}.$$

(3) Find the general solutions of the original equation. Make sure you have considered all cases.

Solution. Note that the integrating factor turns the equation into the exact equation:

$$1+\frac{1}{y}-\frac{x}{y^2}\frac{dy}{dx}=0.$$

But be careful, here we need $y \neq 0$, while the original equation does not require this. In fact we have:

Case 1: $y(x)=0$ is a solution of the original equation.

Case 2: If $y(x) \neq 0$, let $F(x,y)$ be a function satisfying

$$\frac{\partial F}{\partial x}=1+\frac{1}{y}, \qquad \frac{\partial F}{\partial y}=-\frac{x}{y^2}.$$

By the first equality,

$$F(x,y)=\int \left(1+\frac{1}{y}\right)dx+h(y)=x+\frac{x}{y}+h(y),$$

for some function $h(y)$.

It follows from the second equality that

$$-\frac{x}{y^2}=\frac{\partial F}{\partial y}=-\frac{x}{y^2}+h'(y).$$

Thus,

$$h'(y)=0,$$

and we can choose

$$h(y)=0.$$

Therefore,

$$F(x,y)=x+\frac{x}{y}=C$$

are all solutions for $y \neq 0$.

2. (15 points) Solve the initial value problem

$$xy^2 + (1-x)\frac{dy}{dx} = 0, \qquad y(2) = 1,$$

and find the domain of definition of the solution.
Note: The graphs of $x-1$ and e^{3-x} intersect at $x \approx 2.557$.

Solution. Observe that the equation, being equivalent to

$$\frac{x}{1-x} + \frac{1}{y^2}\frac{dy}{dx} = 0$$

away from $x = 1$ and $y = 0$, is separable.
By taking antiderivatives, we have

$$\frac{d}{dx}(-x - \ln|1-x| - y^{-1}) = 0$$

Therefore, the general solutions for the original equation, away from the critical values of x and y are

$$-x - \ln|1-x| - \frac{1}{y} = C,$$

for some constant C.

Now use the initial value $(x_0, y_0) = (2, 1)$. We obtain:

$$C = -2 - \ln 1 - 1 = -3.$$

Thus, the solution to the initial value problem is

$$-x - \ln|1-x| - \frac{1}{y} = -3,$$

or

$$y(x) = (3 - x - \ln|1-x|)^{-1}.$$

Since the initial $x_0 = 2$, using the given information, this solution is valid on $(1, 2.557)$.

3. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^{-1}$ are two solutions of the differential equation

$$x^2y'' + xy' - y = 0,$$

find the general solution of

$$x^2y'' + xy' - y = x^2e^{-x}, \qquad x > 0.$$

Solution. Use the method of variation of parameters.

Let $u_1(x), u_2(x)$ be functions so that $u_1y_1 + u_2y_2$ is a particular solution of the non-homogeneous equation above. Note that

$$W(y_1, y_2) = \det \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -\frac{2}{x},$$

and in our formula

$$g(x) = e^{-x}.$$

Then we have

$$\begin{aligned} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{-x} \\ -x^2e^{-x} \end{pmatrix}. \end{aligned}$$

Thus, we could choose

$$u_1 = -\frac{1}{2}e^{-x},$$

and

$$u_2 = -\frac{1}{2}\int x^2e^{-x}dx.$$

To evaluate the integral, we could use integration by parts. Another approach is by calculating the following three derivatives:

$$(x^2e^{-x})' = 2xe^{-x} - x^2e^{-x} \tag{1}$$

$$(xe^{-x})' = e^{-x} - xe^{-x} \tag{2}$$

$$(e^{-x})' = -e^{-x} \tag{3}$$

It is easy to see that $(1) + 2(2) + 2(3)$ gives

$$(x^2e^{-x} + 2xe^{-x} + 2e^{-x})' = -x^2e^{-x}.$$

Therefore, we could take

$$u_2 = \frac{1}{2}x^2e^{-x} + xe^{-x} + e^{-x},$$

and

$$\begin{aligned} y_p = u_1y_1 + u_2y_2 &= -\frac{1}{2}xe^{-x} + \frac{1}{x}\Big(\frac{1}{2}x^2e^{-x} + xe^{-x} + e^{-x}\Big) \\ &= \Big(1 + \frac{1}{x}\Big)e^{-x}. \end{aligned}$$

We conclude that the general solution of the original equation is

$$y(x) = c_1x + c_2x^{-1} + \Big(1 + \frac{1}{x}\Big)e^{-x},$$

for arbitrary constants c_1, c_2.

4. (20 points) Consider the following *nonlinear* first order differential equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2,$$

where $p(x), q(x), r(x)$ are continuous functions. Equations of this type are called the *Riccati equations*.

(1) Suppose that one solution y_1 is known. Show that the substitution $z = (y - y_1)^{-1}$ transforms the above differential equation (in y) into a *first order linear* ordinary differential equation in the z variable.

Solution. To obtain the first order equation that z satisfies, we take the derivative of z:

$$z' = \Big(\frac{1}{y - y_1}\Big)' = -\frac{y' - y_1'}{(y - y_1)^2}.$$

Now use the assumption that both y_1 and y are solutions, obtaining

$$\begin{aligned}
z' &= -\frac{1}{(y-y_1)^2}[(p(x) + q(x)y + r(x)y^2) - (p(x) + q(x)y_1 + r(x){y_1}^2)] \\
&= -\frac{1}{(y-y_1)^2}[q(x)(y - y_1) + r(x)(y - y_1)(y + y_1)] \\
&= -q(x)z - r(x)\frac{y + y_1}{y - y_1} \\
&= -q(x)z - r(x)\Big(1 + \frac{2y_1}{y - y_1}\Big) \\
&= -q(x)z - r(x) - 2r(x)y_1(x)z.
\end{aligned}$$

To put this in a more concise form:

$$z' = -r - (q + 2ry_1)z.$$

This is clearly a first order linear ODE in z.

(2) Now, use what you obtained in part (1) to find another solution of the differential equation

$$\frac{dy}{dx} = (x^3 + 1) - 2x^2 y + xy^2,$$

given that $y_1(x) = x$ is a solution.

Solution. In this equation,

$$p(x) = x^3 + 1, \quad q(x) = -2x^2, \quad r(x) = x.$$

Therefore, by the previous part and $y_1 = x$,

$$z' = -x - (-2x^2 + 2x^2)z = -x.$$

Hence,

$$z = -\frac{1}{2}x^2.$$

Substituting back to y, it follows that

$$y = \frac{1}{z} + y_1 = -2x^{-2} + x$$

is another solution.

5. (15 points) Suppose that, under natural conditions, the population of fish in a pond obeys the model

$$\dot{p} = p(1-p).$$

Now, suppose that people start harvesting from the pond at a rate of rp $(0 < r < 1)$ per unit time, which is *relative* to the population p.

(1) Establish a model (differential equation) which describes the population of fish in the pond after the harvesting has started.

Solution.

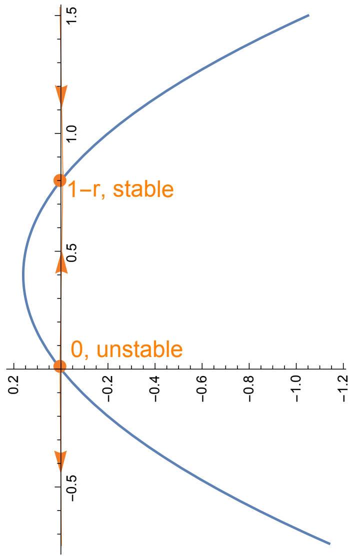
$$\dot{p} = p(1-p) - rp.$$

(2) Sketch the phase line corresponding to the equation that you have written. Specify the equilibria and their stability. Assuming that $p(0) > 0$, is it possible that, for certain values of $r \in (0, 1)$, the fish would go extinct under our harvesting scheme?

Solution. Note that the equation can be rewritten as

$$\dot{p} = p(1-r-p).$$

It is not hard to see that the right hand side is a quadratic function which vanishes at $p = 0$ and $p = 1 - r$. Thus, we have the plot:



In fact, $p = 0$ and $p = 1 - r$ are the two equilibria, which are unstable and stable, respectively.

From the analysis, we can see that there exists no such r so that the fish goes extinct under the harvesting rate rp.

6. (5 points) What is a first order autonomous equation that has exactly the phase line below?

Solution. We look for an equation that looks like

$$y' = f(y).$$

The direction of the arrows tell us the sign of the function $f(y)$ is $+, -, -, +$ on the intervals $(-\infty, -1), (-1, 1), (1, 3), (3, \infty)$, respectively. We know that $f(y)$ vanishes at $-1, 0, 1, 3$. We also know from the two successive negative sign that the graph of the function $f(y)$ touches the point $(1, 0)$ in the y-y' plane but does not pass through the y-axis. So we could try

$$y' = f(y) = (y+1)(y-1)^2(y-3).$$

It is easy to check all the signs and conclude that this is a correct choice. Of course, answers to this question are not unique.