EXAM II, MATH 353 SUMMER I 2015

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test contains **8 pages** and **6 questions**. You have **75 minutes** to answer all the questions. *Good Luck* !

Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \qquad \mathcal{L}\{f'(t)\} = -f(0)+sF(s)$$

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

Question	Max. Points	Score
1	15	
2	15	
3	30	
4	20	
5	15	
6	5	
Total	100	

1. (15 points) Given an appropriate function $f(t)$, use the definition of the Laplace transform to prove the following *scaling formula*:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right),$$

where $a > 0$ is a constant.

Solution. By the definition of the Laplace transform,

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}dt.$$

Now, using the substitution $x = at$, we have

$$\begin{aligned}\mathcal{L}\{f(at)\} &= \frac{1}{a}\int_0^\infty f(x)e^{-\frac{s}{a}x}dx \\ &= \frac{1}{a}F\left(\frac{s}{a}\right).\end{aligned}$$

This completes the proof.

2. (15 points) Using the Laplace transform, find a function $f(x)(x \geq 0)$ that satisfies the integral equation

$$f(x) - \int_0^x e^{x-t} f(t)dt = 1.$$

Solution. Note that this integral equation can be rewritten as

$$f(x) - e^x * f(x) = 1.$$

Taking Laplace transform on both sides gives

$$F(s) - \mathcal{L}\{e^x * f(x)\} = \frac{1}{s},$$

that is,

$$F(s) - \frac{1}{s-1}F(s) = \frac{1}{s},$$

or

$$\frac{s-2}{s-1}F(s) = \frac{1}{s}.$$

Hence,

$$\begin{aligned} F(s) = \frac{s-1}{s(s-2)} &= \frac{1}{s-2} - \frac{1}{s(s-2)} \\ &= \frac{1}{s-2} - \frac{1}{2}\Big(\frac{1}{s-2} - \frac{1}{s}\Big) \\ &= \frac{1}{2}\frac{1}{s-2} + \frac{1}{2s}. \end{aligned}$$

Finally, take the inverse Laplace transform using $\mathcal{L}\{1\} = s^{-1}$ and the first shift theorem, we obtain

$$f(x) = \frac{1}{2}(e^{2x} + 1).$$

3. (30 points) Consider the initial value problem

$$y'' + y = g_k(t), \qquad y(0) = y'(0) = 0,$$

where

$$g_k(t) = \frac{1}{2k}(u_{5-k}(t) - u_{5+k}(t))$$

and $k \in (0, 1]$ is a constant.

(1) Use the Laplace transform to find the solution $y_k(t)$ of the initial value problem above.

Solution. Apply the Laplace transform to both sides of the equation, and we obtain

$$(s^2+1)Y_k(s) = G_k(t) = \frac{1}{2k}(e^{-(5-k)s} - e^{-(5+k)s})\frac{1}{s}.$$

Thus,

$$Y_k(s) = \frac{1}{2k}(e^{-(5-k)s} - e^{-(5+k)s})\frac{1}{s(s^2+1)}.$$

Using partial fractions, let

$$H(s) = \frac{As+B}{s^2+1} + \frac{C}{s} = \frac{1}{s(s^2+1)}.$$

We have $A + C = B = 0$ and $C = 1$.
Hence,

$$H(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

and

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 1 - \cos t.$$

Therefore, by the second shift theorem,

$$\begin{aligned} y_k(t) &= \frac{1}{2k}(u_{5-k}(t)h(t-(5-k)) - u_{5+k}(t)h(t-(5+k))) \\ &= \frac{1}{2k}(u_{5-k}(t)(1-\cos(t-5+k)) - u_{5+k}(t)(1-\cos(t-5-k)). \end{aligned}$$

(2) Calculate $\lim_{k\to 0^+} y_k(t)$. Hint: Consider the cases $0 \le t < 5$, $t = 5$ and $t > 5$.

Solution. For $t < 5$, $\lim_{k\to 0^+} y_k(t) = 0$ because both $u_{5-k}(t)$ and $u_{5+k}(t)$ are zero for small k.
For $t = 5$, $\lim_{k\to 0^+} y_k(t) = 0$ because $u_{5+k}(t) = u_{5+k}(5) = 0$ for all $k > 0$, we have $\lim_{k\to 0^+} y_k(5) = \lim_{k\to 0^+} \frac{1}{2k}(1-\cos k) = \lim_{k\to 0^+} \frac{\sin k}{2} = 0$.
For $t > 5$, as long as k is small enough, we have $5 - k < 5 + k < t$ and $u_{5-k}(t) = u_{5+k}(t) = 1$, hence

$$\begin{aligned} \lim_{k\to 0^+} y_k(t) &= \lim_{k\to 0^+} \frac{1}{2k}(\cos(t-5-k) - \cos(t-5+k)) \\ &= -\frac{d}{dt}\cos(t-5) \\ &= \sin(t-5). \end{aligned}$$

Therefore,

$$\lim_{k\to 0^+} y_k(t) = u_5(t)\sin(t-5).$$

(3) Solve the initial value problem

$$y'' + y = \delta(t-5), \qquad y(0) = y'(0) = 0.$$

Note that the only difference than the previous initial value problem is that the forcing function is changed to a unit impulse at $t_0 = 5$.

Solution. Similarly as in part (1), taking the Laplace transform leads to

$$(s^2+1)Y(s) = e^{-5s}.$$

Therefore,

$$Y(s) = e^{-5s}\frac{1}{s^2+1}.$$

By the second shift theorem, we have

$$y(t) = u_5(t)\sin(t-5).$$

(4) Compare your answers in part (2) and (3). What do you observe? What's your guess about the relation between $\lim_{k\to 0^+} g_k(t) = u_5'(t)$ and $\delta(t-5)$? Justify your guess using the derivative property of the Laplace transform if you can (Optional). Note: If you are curious why $\lim_{k\to 0^+} g_k(t) = u_5'(t)$, note that $u_{5-k}(t) = u_5(t+k)$ and $u_{5+k} = u_5(t-k)$.

Solution. Observe that the answers in (2) and (3) are the same. We guess that in passing to the limit $k \to 0^+$, the forcing function

$$g_k(t) = \frac{1}{2k}(u_{5-k}(t) - u_{5+k}(t)) = \frac{1}{2k}(u_5(t+k) - u_5(t-k))$$

would tend to the impulse function $\delta(t-5)$.
In fact, if we interpret $\lim_{k\to 0^+} g_k(t)$ as $u_5'(t)$, then by the derivative property of the Laplace transform, we have

$$\mathcal{L}\{u_5'(t)\} = s\mathcal{L}\{u_5(s)\} = s\frac{e^{-5s}}{s} = e^{-5s}.$$

Therefore, formally,

$$u_5'(t) = \mathcal{L}^{-1}\{e^{-5s}\} = \delta(t-5).$$

4. (20 points) Use the series method to find a *fundamental set of solutions* for the differential equation

$$(x^3+8)y'' + e^x y' + 2y = 0,$$

about $x = 0$. You only need to calculate terms up to the *third* power of x in each of your solutions. What is a radius of convergence that you could claim about the series solutions, why?

Solution. About the point $x = 0$, let

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Then we have

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots$$

Also note that

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

We have, by the equation,

$$(x^3+8)(2a_2 + 6a_3 x + \dots) + (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots)(a_1 + 2a_2 x + 3a_3 x^2 + \dots)$$
$$+ 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) = 0.$$

Grouping the terms in powers of x and look at their coefficients, we must have

$$16a_2 + a_1 + 2a_0 = 0,$$
$$48a_3 + a_1 + 2a_2 + 2a_1 = 48a_3 + 3a_1 + 2a_2 = 0,$$
$$\dots$$

Therefore,

$$a_2 = -\frac{1}{8}a_0 - \frac{1}{16}a_1,$$
$$a_3 = -\frac{1}{16}a_1 - \frac{1}{24}a_2 = \frac{1}{192}a_0 - \frac{23}{384}a_1.$$

The fundamental solutions are thus

$$y_1(x) = 1 - \frac{1}{8}x^2 + \frac{1}{192}x^3 + \dots, \qquad y_2(x) = x - \frac{1}{16}x^2 - \frac{23}{384}x^3 + \dots$$

For the radius of convergence, note that the equation can be rewritten as

$$y'' + \frac{e^x}{x^3+8}y' + \frac{2}{x^3+8}y = 0.$$

The roots of the polynomial $x^3 + 8$ are the third roots of unity multiplied by -2, hence all at distance 2 to the origin. Therefore, the Taylor expansion at $x = 0$ of the functions

$$\frac{e^x}{x^3+8}, \qquad \frac{2}{x^3+8}$$

are valid with the radius of convergence 2. By a theorem in class, we conclude that the radius of convergence of our series solution is at least 2.

5. (15 points) How many solutions does the two-point boundary problem

$$x^2y'' + xy' + 4y = 0, (x > 0) \qquad y(1) = 0, y(e^\pi) = 0$$

have?
Hint: In looking for the general solutions of the equation itself, try solutions in the form x^r, where r is a complex number.

Solution. Note that this is an Euler equation, and we look for solutions in the form $y(x) = x^r$. Plugging this in the equation, we obtain

$$[r(r-1) + r + 4]x^r = (r^2 + 4)x^r = 0.$$

Hence,

$$r = \pm 2i,$$

and for $r = 2i$ we get the solution

$$x^r = e^{\ln x \cdot 2i} = \cos(2\ln x) + i\sin(2\ln x).$$

Note that the real and imaginary parts of x^r are solutions of the equation, hence we get the general solution

$$y(x) = c_1\cos(2\ln x) + c_2\sin(2\ln x).$$

Now take in to account the boundary values,

$$y(1) = c_1 = 0, \qquad y(e^\pi) = c_1 = 0.$$

Hence,

$$y(x) = c\sin(2\ln x)$$

solves the boundary value problem, and c can be any constant. In particular, we get infinitely many solutions.

6. (5 points) What is the *radius of convergence* of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}(x-1)^n?$$

Solution. Note that the power series is centered at $x_0 = 1$ and

$$a_n = \frac{(-1)^n n^3}{2^n}.$$

By the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{2} \frac{(n+1)^3}{n^3} = \frac{1}{2}.$$

Therefore, the radius of convergence is 2.