EXAM II, MATH 353 SUMMER I 2015

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test contains **8 pages** and **6 questions**. You have **75 minutes** to answer all the questions. *Good Luck* !

Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \qquad \mathcal{L}\{f'(t)\} = -f(0)+sF(s)$$

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

Question	Max. Points	Score
1	15	
2	15	
3	30	
4	20	
5	15	
6	5	
Total	100	

1. (15 points) Given an appropriate function $f(t)$, use the definition of the Laplace transform to prove the following *scaling formula*:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right),$$

where $a > 0$ is a constant.

2. (15 points) Using the Laplace transform, find a function $f(x)(x \geq 0)$ that satisfies the integral equation

$$f(x) - \int_0^x e^{x-t} f(t)dt = 1.$$

3. (30 points) Consider the initial value problem

$$y'' + y = g_k(t), \qquad y(0) = y'(0) = 0,$$

where

$$g_k(t) = \frac{1}{2k}(u_{5-k}(t) - u_{5+k}(t))$$

and $k \in (0, 1]$ is a constant.

(1) Use the Laplace transform to find the solution $y_k(t)$ of the initial value problem above.

(2) Calculate $\lim_{k \to 0^+} y_k(t)$. Hint: Consider the cases $0 \leq t < 5$, $t = 5$ and $t > 5$.

(3) Solve the initial value problem

$$y'' + y = \delta(t-5), \qquad y(0) = y'(0) = 0.$$

Note that the only difference than the previous initial value problem is that the forcing function is changed to a unit impulse at $t_0 = 5$.

(4) Compare your answers in part (2) and (3). What do you observe? What's your guess about the relation between $\lim_{k\to 0^+} g_k(t) = u_5'(t)$ and $\delta(t-5)$? Justify your guess using the *derivative property* of the Laplace transform if you can (Optional). Note: If you are curious why $\lim_{k\to 0^+} g_k(t) = u_5'(t)$, note that $u_{5-k}(t) = u_5(t+k)$ and $u_{5+k} = u_5(t-k)$ $(k < 5)$.

4. (20 points) Use the series method to find a *fundamental set of solutions* for the differential equation

$$(x^3 + 8)y'' + e^x y' + 2y = 0,$$

about $x = 0$. You only need to calculate terms up to the *third* power of x in each of your solutions. What is a radius of convergence that you could claim about the series solutions, why?

5. (15 points) How many solutions does the two-point boundary problem

$$x^2y'' + xy' + 4y = 0, (x > 0) \qquad y(1) = 0, y(e^\pi) = 0$$

have?
Hint: In looking for the general solutions of the equation itself, try solutions in the form x^r, where r is a complex number.

6. (5 points) What is the *radius of convergence* of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}(x-1)^n?$$