EXAM I, MATH 353 SUMMER I 2015

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test contains **9 pages** and **6 questions**. You have **75 minutes** to answer all the questions.
Good Luck !

Useful Formulas:

$$y_p(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(s)g(s)}{W(y_1,y_2)}ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)g(s)}{W(y_1,y_2)}ds,$$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

You are responsible for identifying correctly the situation in which these formulas can be applied.

Question	Max. Points	Score
1	25	
2	15	
3	20	
4	20	
5	15	
6	5	
Total	100	

1. (25 points) Consider the differential equation

$$y^2 + y - x\frac{dy}{dx} = 0.$$

(1) Show that this equation is *not exact.*

(2) Find an integrating factor for this equation.
Hint: There exists an integrating factor $v(y)$. Also, before proceeding to the next question, you may want to check that the original equation becomes exact when multiplied by the integrating factor you obtained.

(3) Find the general solutions of the original equation. Make sure you have considered all cases.

2. (15 points) Solve the initial value problem

$$xy^2 + (1-x)\frac{dy}{dx} = 0, \qquad y(2) = 1,$$

and find the domain of definition of the solution.
Note: The graphs of $x-1$ and e^{3-x} intersect at $x \approx 2.557$.

3. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^{-1}$ are two solutions of the differential equation

$$x^2y'' + xy' - y = 0,$$

find the general solution of

$$x^2y'' + xy' - y = x^2e^{-x}, \qquad x > 0.$$

4. (20 points) Consider the following *nonlinear* first order differential equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2,$$

where $p(x), q(x), r(x)$ are continuous functions. Equations of this type are called the *Riccati equations*.

(1) Suppose that one solution y_1 is known. Show that the substitution $z = (y - y_1)^{-1}$ transforms the above differential equation (in y) into a *first order linear* ordinary differential equation in the z variable.

(2) Now, use what you obtained in part (1) to find another solution of the differential equation

$$\frac{dy}{dx} = (x^3 + 1) - 2x^2 y + xy^2,$$

given that $y_1(x) = x$ is a solution.

5. (15 points) Suppose that, under natural conditions, the population of fish in a pond obeys the model

$$\dot{p} = p(1 - p).$$

Now, suppose that people start harvesting from the pond at a rate of rp $(0 < r < 1)$ per unit time, which is *relative* to the population p.

(1) Establish a model (differential equation) which describes the population of fish in the pond after the harvesting has started.

(2) Sketch the phase line corresponding to the equation that you have written. Specify the equilibria and their stability. Assuming that $p(0) > 0$, is it possible that, for certain values of $r \in (0, 1)$, the fish would go extinct under our harvesting scheme?

6. (5 points) What is a first order autonomous equation that has exactly the phase line below?