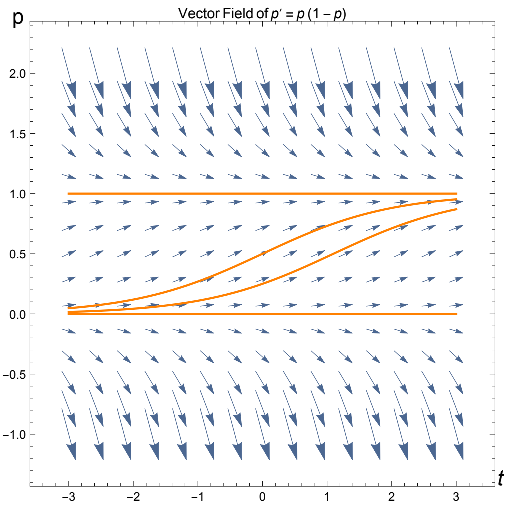
MATH 3430-03 SPRING 2020
AUTONOMOUS EQUATIONS

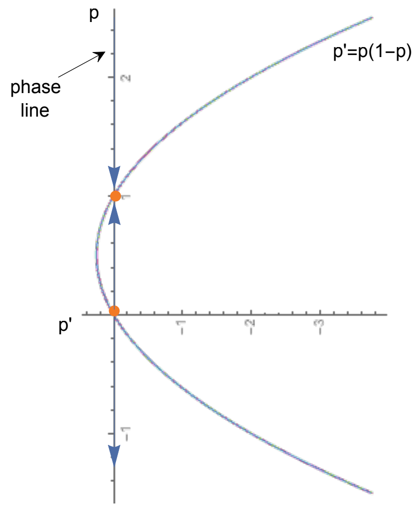
Definition. A first-order ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(y)$$

is called **autonomous**.

It is clear that the direction field of an autonomous equation is invariant under translation in the x-direction.





Logistic Model. A plot of the direction field of the standard *logistic equation*,

$$\dot{p} = p(1-p),$$

is shown above, together with the graph of the function $g(p) = p(1-p)$. There I have rotated the graph of $g(p) = p(1-p)$ counter-clockwise by $90°$ so that the p axis goes in parallel with that in the vector field plot. In doing this, it is easy to see that the p-intercept of the graph of $g(p) = p(1-p)$ corresponds to the equilibrium solutions (i.e., solutions that are constant functions) of the ODE.

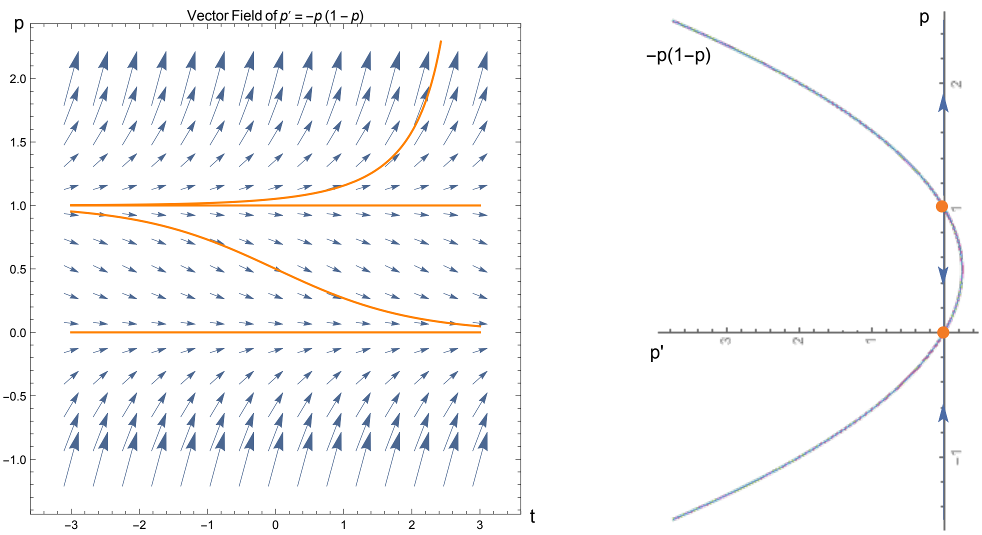
Moreover, we see that when p is between $(0, 1)$, $\dot{p}$ remains positive. This means that all the arrows in the direction field have positive tangent as long as $p \in (0, 1)$. To capture this property, we put a upward arrow between $(0, 1)$ along the p-axis (phase line). For a similar reason, we put a downward arrow within the interval $(1, \infty)$ along the p-axis, to indicate that when p takes value in this interval, arrows in the direction field have negative tangent. For a same reason, we put a downward arrow in the interval $(-\infty, 0)$ along the p-axis. The p-axis in the graph on the right, with all the arrows and equilibriums marked, is called a **phase line plot** corresponding to the logistic equation $\dot{p} = p(1-p)$. Now, convince yourself that these arrows are capable of telling us the stability of all the equilibria. *(Do you see that along the phase line, nearby the equilibrium $p = 1$, arrows* *are pointing towards it? How about $p = 0$? What is your conclusion?)*

One of the reasons why we like to use phase lines to do analysis is their simplicity: (i) for an autonomous equation $\dot{y} = F(y)$, the y intercept, if any, of the graph of $F(y)$ gives us the equilibria of the system; (ii) by figuring out the sign of $F(y)$ in each interval separated by the equilibria, we can draw arrows along these intervals as we did before, thus being able to tell whether an equilibrium is stable or not.

Threshold Model. Consider the following scenario: when the population is below a certain level, for certain species do not experience exponential growth, instead, the population can no longer support itself and thus goes to extinction. This scenario may be modeled by the following ODE:

$$\dot{p} = -rp\left(1 - \frac{p}{T}\right).$$

This equation differs from the general logistic equation only by a negative sign on the right hand side. So if we plot the graph of $F(p)$ in this case, it looks exactly like the graph of $F(p)$ in the logistic model being flipped over the p-axis. Therefore, when $r = T = 1$, we obtain the phase line simply by reversing all the arrows in the phase line of the logistic equation. In the plot below, the equilibrium $p = 0$ is stable and $p = 1$ is not.



Remark. The phase line analysis has limitations. For instance, in the standard threshold model with $r = T = 1$, the phase line tells us that solutions would increase if we set an initial value to be above 1. You may think that the solution would then increase for all x. However, this is not the case. By directly solving the equation (separation of variables), we'll notice that such solutions goes unbounded in finite time! This is something that we are not able to tell from the phase line alone.

Threshold & Environmental Cap: Combination. Suppose that we want to design a population model that involves both an extinction threshold T and an environmental cap R. We start with a phase line marked with $0, T, R$ $(0 < T < R)$.

For the model to work, we want that, for an initial condition that belongs to the intervals $(0, T)$, (T, R), and (R, ∞), respectively, the solutions need to approach zero, increase to R, and decrease to R. This tells us which arrows to put in these intervals: downward for $(0,T)$ and (R,∞), upward for (T,R).

Now, we are looking for a function $F(p)$ that vanishes at 0, T and R. A natural choice would be

$$F(p) = \lambda p \left(1 - \frac{p}{T}\right)\left(1 - \frac{p}{R}\right),$$

for some constant λ. In addition, we need that when $p > R$, $F(p) < 0$. Thus $\lambda < 0$. Let $-\lambda = r$, where $r > 0$. As a result, we have obtained the model:

$$\dot{p} = -rp\left(1 - \frac{p}{T}\right)\left(1 - \frac{p}{R}\right).$$

To check that this model works as expected, we plot the direction field as well as the phase line below in the case when $r = T = 1$ and $R = 3$.