Math 3430-03 Spring 2020
Practice Exam I

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the CU Community Standard.

Name: Solutions Signature: ______________________

Instructions:

- You may use a formula sheet, A4 size, front and back, prepared on your own. Otherwise, notes, books, calculators or computers are not allowed in this exam.
- Make sure you show the work that leads to your answer to receive full credit. If you are using a theorem or a fact to draw some conclusions, quote the result.
- This test contains **5 questions**. You have **50 minutes** to answer all the questions.

Good Luck !

Question	Score
1	/ 20
2	/ 20
3	/ 20
4	/ 20
5	/ 20
Total	**/100**

1. First-order ODEs

1. Solve the following 1st-order initial value problem:

$$\underbrace{(y\cos x + e^{2y})}_{P} + \underbrace{(\sin x + 2xe^{2y} + 2y)}_{Q}\frac{dy}{dx} = 0, \qquad y(0) = 1.$$

$$\left.\begin{aligned} P_y &= \cos x + 2e^{2y} \\ Q_x &= \cos x + 2e^{2y} \end{aligned}\right\} \text{equal} \quad \therefore \quad \text{ODE is exact.}$$

$$F(x,y) = \int P\,dx = y\sin x + xe^{2y} + g(y)$$

$$F(x,y) = \int Q\,dy = y\sin x + xe^{2y} + y^2 + h(x).$$

$$\Rightarrow \quad \text{can take} \quad F(x,y) = y\sin x + xe^{2y} + y^2$$

General sol$^{\underline{n}}$: $\boxed{y\sin x + xe^{2y} + y^2 = C}$ || I.C. $\Rightarrow$ $\boxed{C = 1}$

2. Find the general solutions of the following 1st-order ODE:

$$x\frac{dy}{dx} - 3y = x^4.$$

linear 1st order ODE $\rightsquigarrow$ standard form:

$$y' \underbrace{- \frac{3}{x}}_{P} y = \underbrace{x^3}_{Q}.$$

$$\therefore \quad \mu = e^{\int p(x)\,dx} = e^{\int -\frac{3}{x}} = e^{-3\ln|x| + c}$$

$$\underset{(\text{choose})}{=} \frac{1}{x^3}$$

$$\therefore \quad y = \frac{1}{\mu}\int \mu Q\,dx = x^3 \cdot \left(\int \frac{1}{x^3}\cdot x^3\right)$$

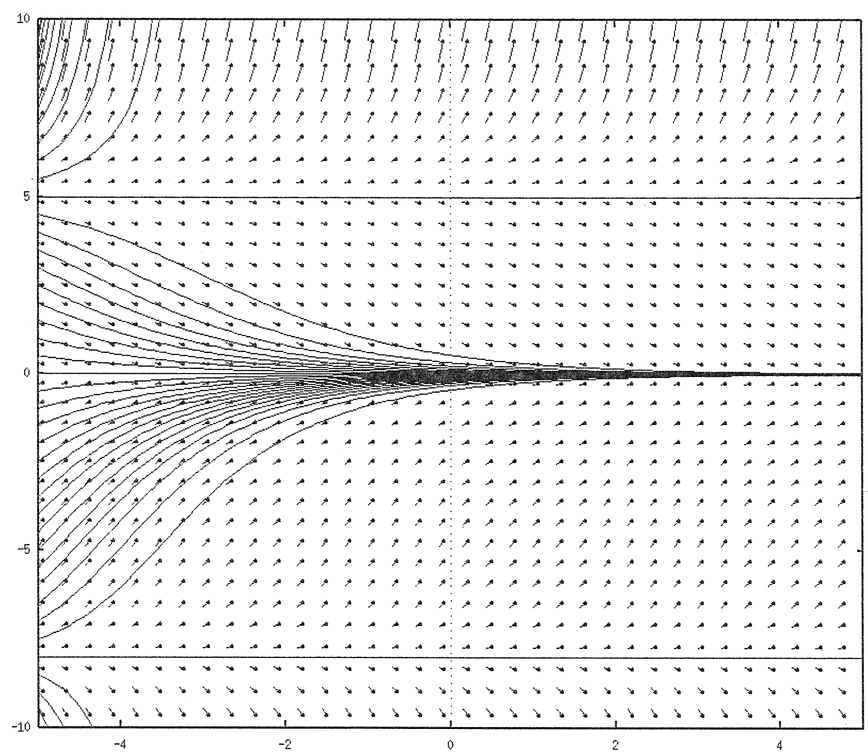
$$= x^3(x + C).$$

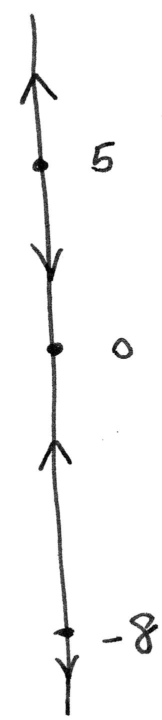
3. This question is concerned with "direction fields".

(1) What's special about the direction field of an **autonomous** first-order ODE?

It's invariant along the x-direction.

(2) Write down a 1st-order ODE $y' = f(y)$ that has the following direction field (with various integral curves). (Note: The three horizontal integral curves are at $y = -8, 0, 5$, respectively.)





$$y' = y(y+8)(y-5).$$

(coeff. is >0 since $y'>0$ when $y>5$.)

4. Find a **basis** for the solution space of the following 6th-order ODE:

$$y^{(6)} + 7y^{(5)} - 6y^{(4)} - 70y''' - 235y'' - 309y' - 252y = 0$$

You may find this formula useful:

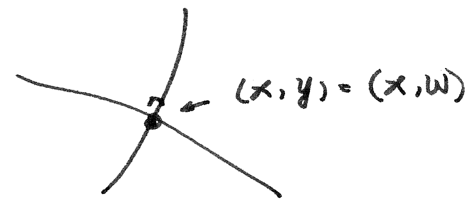
$$\lambda^6 + 7\lambda^5 - 6\lambda^4 - 70\lambda^3 - 235\lambda^2 - 309\lambda - 252$$
$$= (\lambda^2 + 2\lambda + 3)^2(\lambda + 7)(\lambda - 4).$$

root r	multiplicity m
$-1+\sqrt{2}i$	2
$-1-\sqrt{2}i$	2
-7	1
4	1

Basis of solution

$$\left\{ e^{-x}\cos\sqrt{2}x,\ e^{-x}\sin\sqrt{2}x,\ xe^{-x}\cos\sqrt{2}x,\ xe^{-x}\sin\sqrt{2}x,\ e^{-7x},\ e^{4x} \right\}.$$

5. Let C_1 be a 1-parameter family of curves that satisfy the equation
$$y^3 = cx^5,$$
where c is the parameter. Now establish a differential equation for its orthogonal trajectories C_2 and then solve the equation.



$$3y^2y' = 5cx^4$$

$$\Rightarrow \quad y' = \frac{5c}{3y^2}x^4$$

$$\Rightarrow \quad w' = -\frac{3y^2}{5cx^4} = -\frac{3x}{5y} = -\frac{3x}{5w}$$

orthogonal traj.

using $c = y^3/x^5$

using: @ intersection, $y = w$.

$$\therefore \quad 5ww' = -3x$$

$\downarrow$ separable eq^{n}.

$$\frac{5}{2}w^2 = -\frac{3}{2}x^2 + C$$

or

$$\boxed{5w^2 + 3x^2 = C'.}$$

arbitrary constant (positive.)