Math 3430-03 Spring 2020
Exam I

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the CU Community Standard.

Name: Solutions Signature: ____________

Instructions:

- You may use a formula sheet, A4 size, front and back, prepared on your own. Otherwise, notes, books, calculators or computers are not allowed in this exam.
- Make sure you show the work that leads to your answer to receive full credit. If you are using a theorem or a fact to draw some conclusions, quote the result.
- This test contains **5 questions**. You have **50 minutes** to answer all the questions.

Good Luck !

Question	Score
1	/ 20
2	/ 20
3	/ 20
4	/ 20
5	/ 20
Total	**/100**

1. First-order ODEs

1. Solve the following 1st-order initial value problem:

$$\underbrace{(y^2e^{xy^2}+3x^2)}_{P}+\underbrace{(2xye^{xy^2}+\cos y)}_{Q}\frac{\mathrm{d}y}{\mathrm{d}x}=0, \qquad y(0)=\frac{\pi}{2}.$$

$$\left.\begin{aligned} P_y &= 2ye^{xy^2}+2xy^3e^{xy^2} \\ Q_x &= 2ye^{xy^2}+2xy^3e^{xy^2} \end{aligned}\right\}\text{ equal} \qquad \therefore \text{ Equation is exact.}$$

$$F(x,y)=\int P\,dx = e^{xy^2}+x^3+g(y)$$

$$F(x,y)=\int Q\,dy = e^{xy^2}+\sin y+h(x)$$

$$\therefore \quad F = e^{xy^2}+x^3+\sin y$$

General solution: $\boxed{e^{xy^2}+x^3+\sin y = C}$

IC $\Rightarrow$ $1+0+\sin\frac{\pi}{2}=C$ $\Rightarrow$ $\boxed{C=2}$

2. Find the general solutions of the following ODE:

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\frac{\sin x}{x}, \qquad x\neq 0.$$

Equation in standard form (linear)

$$y'+\underbrace{\frac{2}{x}}_{P}y=\underbrace{\frac{\sin x}{x^2}}_{Q}$$

$\therefore$ Integrating factor $\mu = e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2.$

$\therefore$ general solution

$$y=\frac{1}{\mu}\int \mu Q\,dx = \frac{1}{x^2}\int \sin x\,dx$$

$$\Rightarrow \quad \boxed{y=\frac{1}{x^2}(-\cos x+C)}$$

3. This question is concerned with 1st order homogeneous equations.

(1) A 1st-order ODE of the form

$$y' = f(x, y),$$

is said to be **homogeneous** if $f(x, y)$ is *homogeneous of order* 0. What does "homogeneous of order 0" mean?

for any $\lambda > 0$, $\quad f(\lambda x, \lambda y) = f(x, y)$.

(2) What's special about the direction field of a **homogeneous** first-order ODE?

it's constant along rays through the origin.

or, straight lines

(3) Solve the following 1st-order initial value problem and specify the open interval (a, b) on which your solution is defined.

$$(xy + y^2) - x^2\frac{dy}{dx} = 0, \qquad y(1) = 1.$$

$$y' = \frac{xy + y^2}{x^2} \qquad \text{homogeneous}$$

let $y = ux$, so u satisfies (from class)

$$\frac{du}{(u+u^2) - u} = \frac{dx}{x}$$

$$\therefore \quad \frac{du}{u^2} = \frac{dx}{x} \Rightarrow -\frac{1}{u} = \ln|x| + C.$$

This is just $\quad -\frac{x}{y} = \ln|x| + C$, or $\quad y = -\frac{x}{\ln|x| + C}$

I.C at $(1,1) \Rightarrow x > 0$ and $C = -1$

$$\therefore \quad \boxed{y = -\frac{x}{(\ln x) - 1} \quad \text{and} \quad x \in (0, e).}$$

since $\ln x - 1 \neq 0$.

4. Find a **basis** for the solution space of the following 6th-order ODE:

$$y^{(6)} - 12y^{(5)} + 35y^{(4)} + 31y''' - 90y'' - 175y' - 750y = 0$$

You may find this formula useful:

$$\begin{aligned} &\lambda^6 - 12\lambda^5 + 35\lambda^4 + 31\lambda^3 - 90\lambda^2 - 175\lambda - 750 \\ &= (\lambda^2 + \lambda + 3)(\lambda - 5)^3(\lambda + 2). \end{aligned}$$

λ root	multiplicity
$\frac{-1+\sqrt{11}\,i}{2}$	1
$\frac{-1-\sqrt{11}\,i}{2}$	1
5	3
-2	1

$\therefore$ Basis of solutions:

$$\left\{ e^{-\frac{x}{2}}\cos\frac{\sqrt{11}}{2}x,\ e^{-\frac{x}{2}}\sin\frac{\sqrt{11}}{2}x,\ e^{5x},\ xe^{5x},\ x^2e^{5x},\ e^{-2x} \right\}.$$

5. Let C_1 be a 1-parameter family of curves that satisfy the equation

$$\sin y = ce^{x^2},$$

where c is the parameter. Now establish a differential equation for its orthogonal trajectories C_2 and then solve the equation.

$$\sin y = ce^{x^2} \xrightarrow[\text{sides}]{\frac{d}{dx} \text{ on both}} (\cos y)\, y' = \underline{c} \cdot 2x \underline{e^{x^2}}$$

$$= 2x \cdot \sin y$$

now $$y' = (\tan y) \cdot (2x)$$

So $$w' = -\frac{1}{y'} = -(\cot y) \cdot \frac{1}{2x}$$

$$= -(\cot w) \cdot \frac{1}{2x}$$

In other words:

$$-(\tan w)\, w' = \frac{1}{2x}$$

$$\downarrow \text{ separable eq}^{\text{n}}.$$

$$\ln|\cos w| = \frac{1}{2} \ln|x| + C$$

$$\therefore \quad \boxed{\cos^2 w = A \cdot x \qquad A: \text{constant}}$$

orthogonal trajectory of C_1