LAPLACE TRANSFORM I

1. Definition and Questions

The definition of the Laplace transform is simple: For an appropriate function $f(t)$, the *Laplace transform* of $f(t)$ is a function $F(s)$ which is equal to

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

Naturally, just by looking at this expression, one might ask:

- What are the domain and range of this transformation?
- Is the transformation one-to-one? That is, does $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ imply $f(t)=g(t)$?
- Is the transformation linear?
- What does $\int_0^\infty$ mean?
- What is an application of Laplace transformation?

Let's answer the first question:

What are the domain and range of the Laplace transformation?

To be more explicit, we are really asking:

For which functions $f(t)$, and for which values of s is the integral

$$\int_0^\infty f(t)e^{-st}dt$$

defined?

In fact, this improper integral is by definition the limit

$$\lim_{A\to\infty} \int_0^A f(t)e^{-st}dt.$$

Thus, we are asking, for which functions $f(t)$ does this limit exist? Before a characterization of such functions, let's see some examples of the Laplace transform.

2. Examples

2.1. $\mathcal{L}\{1\}$.

By the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}\{1\} &= \lim_{A\to\infty} \int_0^A 1 \cdot e^{-st}dt \\
&= \lim_{A\to\infty} \Big(-\frac{1}{s}e^{-st}\Big|_{t=0}^{A}\Big) \\
&= \lim_{A\to\infty} \Big(-\frac{1}{s}e^{-sA} + \frac{1}{s}\Big).
\end{aligned}$$

This limit converges to s^{-1} for $s > 0$. Hence,

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

2.2. $\mathcal{L}\{t\}$. By the definition,

$$\begin{aligned}\mathcal{L}\{t\} &= \lim_{A\to\infty}\int_0^N te^{-st}dt\\ &= \lim_{A\to\infty} -\frac{1}{s}\Big[te^{-st}\Big|_{t=0}^{A} - \int_0^A e^{-st}dt\Big]\\ &= -\frac{1}{s}\lim_{A\to\infty}(Ae^{-sA}) + \frac{1}{s}\mathcal{L}\{1\}\end{aligned}$$

For $s > 0$, the limit exists and we have

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > 0.$$

2.3. $\mathcal{L}\{t^n\}$. Again,

$$\begin{aligned}\mathcal{L}\{t^n\} &= \lim_{A\to\infty}\int_0^A t^n e^{-st}dt\\ &= \lim_{A\to\infty} -\frac{1}{s}\Big[t^n e^{-st}\Big|_{t=0}^{A} - \int_0^A nt^{n-1}e^{-st}dt\Big]\\ &= -\frac{1}{s}\lim_{A\to\infty}(A^n e^{-sA}) + \frac{n}{s}\mathcal{L}\{t^{n-1}\}.\end{aligned}$$

For $s > 0$, by the L'Hôpital's rule, we have $\lim_{A\to\infty}(A^n e^{-sA}) = 0$. Therefore,

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \qquad s > 0.$$

By induction,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^{n+1}}, \qquad s > 0.$$

2.4. $\mathcal{L}\{\cos t\}$. By definition,

$$\mathcal{L}\{\cos t\} = \lim_{A\to\infty}\int_0^A (\cos t)e^{-st}dt$$

Now we need to evaluate the integral $\int(\cos t)e^{-st}dt$. We approach this by calculating the derivatives of $(\cos t)e^{-st}, (\sin t)e^{-st}$ with respect to t:

$$(e^{-st}\cos t)' = -e^{-st}\sin t - se^{-st}\cos t, \tag{1}$$

$$(e^{-st}\sin t)' = e^{-st}\cos t - se^{-st}\sin t. \tag{2}$$

So, $(2) - s\cdot(1)$ gives

$$(e^{-st}\sin t - se^{-st}\cos t)' = (1+s^2)e^{-st}\cos t.$$

Therefore,

$$\begin{aligned}\int_0^A e^{-st}\cos t\ dt &= \frac{1}{1+s^2}(\sin t - s\cos t)e^{-st}\Big|_{t=0}^{A}\\ &= \frac{e^{-sA}}{1+s^2}(\sin A - s\cos A) + \frac{s}{1+s^2}.\end{aligned}$$

For $s > 0$, taking $A \to \infty$ gives

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}, \qquad s > 0.$$

Remark. We will see a simpler way to find the Laplace transform of $\cos at$ and $\sin at$ once we have the *first shift theorem.*

2.5. $\mathcal{L}\{u_c(t)\}$**.** Here

$$u_c(t) = \begin{cases} 0, & t \in [0, c) \\ 1, & t \in [c, \infty) \end{cases}$$

is called the *Heaviside's step function*, where c is a non-negative constant.

Again, we encounter the integral

$$\int_0^A u_c(t) e^{-st} dt.$$

For A begin sufficiently large, this integral consists of two parts: integration on $[0, c)$ where $u_c(t) = 0$ and integration on $[c, A]$ where $u_c(t) = 1$. Therefore,

$$\mathcal{L}\{u_c(t)\} = \lim_{A\to\infty} \int_0^A u_c(t)e^{-st}dt = \lim_{A\to\infty} \int_c^A e^{-st}dt = \frac{1}{s}e^{-sc}, \qquad s > 0.$$

3. The "Appropriate" functions

To which functions can one apply the Laplace transform?

First of all, we need the integral $\int_0^A f(t)dt$ to make sense, i.e., $f(t)$ is integrable on the intervals $[0, A]$. The function $f(t)$ being continuous certainly guarantees this. However, we can also allow a finite number of discontinuities, say $0 < x_1 < x_2 < ... < x_k < N$, so the integral is the sum of the integrals of $f(t)$ on each of the intervals $[0, x_1), [x_1, x_2), \ldots, [x_k, N]$ where $f(t)$ is continuous. Such functions are called *piecewise continuous* on the interval $[0, N]$. A function is said to be *piecewise continuous on* $[0, \infty)$ if it is piecewise continuous on any $[0, N]$, $N > 0$.

Secondly, we need the limit $\lim_{A\to\infty} \int_0^A f(t)e^{-st}dt$ to exist. Intuitively, fixing s, this is saying that the signed area of the region enclosed by the graph of $f(t)e^{-st}$ and the x-axis is finite. One implication is that $f(t)$ should not increase too fast. For instance, for $s = 1$ and $f(t) = e^t$, we have $f(t)e^{-st} = 1$ and the limit of the integral does not exist. Also note that for the same $f(t)$, the limit of the integral exists for $s > 1$. More generally, if $f(t) = Me^{\alpha t}$ for some $M, \alpha > 0$, then the limit

$$\lim_{N\to\infty} \int_0^N Me^{\alpha t}e^{-st}dt = \lim_{N\to\infty} \int_0^N Me^{(\alpha-s)t}dt = \lim_{N\to\infty} \frac{M}{\alpha - s}(e^{(\alpha-s)N} - 1)$$

exists for all $s > \alpha$. What is nice about this fact is that it actually brings a good amount of "appropriate functions" into our sight, based on the comparison theorem below.

Theorem. If $f(x)$ and $g(x)$ are piecewise continuous functions defined on $[0, \infty)$ and $|f(x)| \leq g(x)$, then the convergence of $\int_0^\infty g(x)dx$ implies the convergence of $\int_0^\infty f(x)dx$.

The following corollary of the theorem tells us which functions are "appropriate" to be put in the Laplace transform.

Corollary. Based on the theorem, we can say that if (a) $f(t)$ *is piecewise continuous on* $[0, N]$ *for any* $N > 0$ and (b) *it is of the exponential order*, then $\int_0^\infty f(t)e^{-st}dt$ converges on the interval (α, ∞) for some $\alpha > 0$.

Note: A function $f(t)$ is said to be of the *exponential order* if there exist constants $M, \alpha > 0$ such that $|f(t)| \leq Me^{\alpha t}$ on $[0, \infty)$.

From now on, we confine ourselves to the category of functions which satisfy the conditions in the corollary above and call these functions "appropriate". As an exercise, you can check that all the $f(t)$'s in the previous section are appropriate.

4. Properties of the Laplace Transform

4.1. **Linearity.** The Laplace transform is linear. In fact, for $f(t), g(t)$ appropriate functions and $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$, both defined for $s \in (\alpha, \infty)$, and any constants a, b, we have that

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty (af(t) + bg(t))e^{-st}dt \\ &= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.\end{aligned}$$

is defined for $s \in (\alpha, \infty)$. This proves the linearity.

4.2. **First shift theorem.** If $f(t)$ is appropriate and $F(s) = \mathcal{L}\{f(t)\}$ is defined for $s \in (\alpha, \infty)$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

is defined for $s \in (\alpha + a, \infty)$.
Proof. By definition,

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{at}f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}dt \\ &= F(s - a).\end{aligned}$$

Since we know that $F(s)$ is defined for $s \in (\alpha, \infty)$, $F(s-a)$ is defined for $s \in (\alpha + a, \infty)$.

Remark. The first shift theorem works for complex-valued a. For instance, letting $a = i$, we have

$$\mathcal{L}\{e^{at} \cdot 1\} = \frac{1}{s - a} = \frac{s + i}{s^2 + 1}.$$

On the other hand, by linearity,

$$\mathcal{L}\{e^{it}\} = \mathcal{L}\{\cos t\} + i\mathcal{L}\{\sin t\}.$$

Hence, comparing the real and imaginary parts, we have

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1}, \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}.$$

4.3. **Second shift theorem.** If $f(t)$ is appropriate and $F(s) = \mathcal{L}\{f(t)\}$ is defined for $s \in (\alpha, 0)$, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > \alpha.$$

Proof. First, you may want to convince yourself that the function $u_c(t)f(t-c)$ is indeed the result of shifting $f(t)$ to the right by c.

By definition,

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty u_c(t)f(t-c)e^{-st}dt \\ &= \int_c^\infty f(t-c)e^{-st}dt \\ &= \int_0^\infty f(\xi)e^{-s(\xi+c)d\xi} \qquad (\xi = t-c) \\ &= e^{-sc}\int_0^\infty f(\xi)e^{-s\xi}d\xi \\ &= e^{-sc}F(s), \qquad s > \alpha.\end{aligned}$$

4.4. **Relation to the derivatives.**

4.4.1. $\mathcal{L}\{f'(t)\}$. Before stating the conditions on $f(t)$, let us calculate formally the Laplace transform of $f'(t)$:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt.$$

Now, using integration by parts (formally), we obtain

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= f(t)e^{-st}\Big|_{t=0}^{\infty} - (-s)\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}.\end{aligned}$$

Back to the question: what are the conditions on the function $f(t)$? First, the result above involves the Laplace transform of $f(t)$ and $f'(t)$, so it is reasonable to have both $f(t), f'(t)$ as being appropriate, i.e., piecewise continuous and of the exponential order. Second, by the fundamental theorem of calculus, the integration by parts only applies to functions that are continuous. You may wonder, isn't $f(t)$ continuous if $f'(t)$ exists? Remember, in our context, finitely many discontinuities can exist and $f'(t)$ may only exist piece-wisely. So, we enforce another condition, that is, $f(t)$ is continuous. To summarize, we state this as a proposition:

Proposition. If $f(t)$ is continuous and $f(t), f'(t)$ are appropriate, i.e., piecewise continuous and of the exponential order, then

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}, \qquad s > \alpha$$

for some $\alpha > 0$.

4.4.2. $\mathcal{L}\{f^{(n)}(t)\}$. The previous derivative property can be applied successively, and the result generalizes to higher derivatives.

Proposition. If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous and if $f(t), \ldots, f^{(n)}(t)$ are appropriate, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0), \qquad s > \alpha$$

for some $\alpha > 0$.

Note, in particular, that when $n = 2$, this is just

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0), \qquad s > \alpha.$$

4.4.3. $\mathcal{L}\{tf(t)\}$. Let us calculate $\mathcal{L}\{tf(t)\}$ formally by the definition

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= \int_0^\infty tf(t)e^{-st}dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\ &= -\frac{d}{ds}\mathcal{L}\{f(t)\} \\ &= -\frac{d}{ds}F(s).\end{aligned}$$

Here we have implicitly interchanged the order of integral and derivative. Particularly, assuming the integral to be absolutely convergent ensures this. Of course, the result above can be extended to

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}.$$

As an exercise, you can prove this formula, then use it to find $\mathcal{L}\{t^n\}$.

5. Inverse Laplace Transform

At the beginning of this notes, we asked the question: *Is the Laplace transform one-to-one?* In other words, could two different functions share the same Laplace transform? An answer to this question is given by the following theorem.

Theorem.(Lerch) For a function $F(s)$, the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, if it exists, is unique in the sense that we allow a difference of function values on *a set that has zero Lebesgue measure* (meaning: a subset of $\mathbb{R}$ that is negligible for integrals).

For example, since we have $\mathcal{L}\{1\} = s^{-1}$, by the theorem, we know that $\mathcal{L}^{-1}\{s^{-1}\} = 1$ and that the 1 on the right hand side of this equality is a *representative* of all the functions that has the Laplace transform being s^{-1}, which are equal to $f(t) = 1$ except on negligible sets.

In practice, how can one figure out the inverse Laplace transform of a function? Given what we know so far, one can look at the function and ask: (1) Is it the Laplace transform of some familiar function? (2) If not, does it appear like the Laplace transform of the derivative/shift/t-multiple of a familiar function? For example, consider the function

$$F(s) = \frac{1}{s^2 - 2s + 2}.$$

If we rewrite it as

$$F(s) = \frac{1}{(s-1)^2 + 1},$$

we recognize that if we have a function $g(t)$ whose Laplace transform is

$$G(s) = \frac{1}{1 + s^2},$$

then $F(s)$ is $G(s)$ shifted to the right by 1, and by the first shift theorem, $f(t)$ is simply $e^t g(t)$. By the list of Laplace transforms that we've computed, we have that $g(t) = \sin t$; hence,

$$f(t) = e^t \sin t.$$

6. Solving Initial Value Problems

In this section, we apply the Laplace transform to solving differential equations. The idea is simple: After a Laplace transform, certain *differential equations* for $y(t)$ get transformed into *algebraic equations* for $F(s)$ whose solutions can be found using algebra. Once we've found $F(s)$, we apply the inverse Laplace transform to get $y(t)$. This is summarized in the diagram below:

$$\begin{tikzcd}[column sep=large, row sep=large]
\boxed{\text{Differential Equation in } y(t)} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \boxed{\text{Algebraic Equation in } \mathcal{L}\{y(t)\}} \arrow[d] \\
\boxed{y(t)} & \boxed{\mathcal{L}\{y(t)\} = F(s)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

Figure 1. Solving a Differential Equation Using the Laplace Transform

Example 1. We apply the Laplace transform to solve the initial value problem:

$$y'' - y' - 2y = 0, \qquad y(0) = 1, y'(0) = 0.$$

First, transform on both sides of the equation and use the initial conditions, we obtain:

$$\begin{aligned}
\mathcal{L}\{y'' - y' - 2y\} &= (s^2F(s) - sy(0) - y'(0)) - (sF(s) - y(0)) - 2F(s) \\
&= (s^2 - s - 2)F(s) - s + 1, \\
\mathcal{L}\{0\} &= 0.
\end{aligned}$$

Hence,

$$F(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Note that we are in a situation of using partial fractions; in fact, let

$$\frac{A}{s-2} + \frac{B}{s+1} = \frac{s-1}{(s-2)(s+1)}.$$

This gives us

$$A(s+1) + B(s-2) = (A+B)s + (A-2B) = s-1.$$

Comparing the coefficients of s gives that

$$A + B = 1, \qquad A - 2B = -1.$$

We obtain

$$A = \frac{1}{3}, \qquad B = \frac{2}{3}.$$

Thus,

$$F(s) = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}.$$

By the linearity of the Laplace transform, first shift theorem, and the fact that $\mathcal{L}\{1\} = s^{-1}$, we obtain:

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t},$$

which is the solution of the initial value problem.

As we can see, this procedure is quite "algorithmic". However, it may sometimes require a bit of thought, and experience, to cope with the last step: finding the inverse Laplace transform of $F(s)$. Here is an example in which $\mathcal{L}^{-1}\{F(s)\}$ may not look obvious at first sight.

Example 2. We apply the Laplace transform to solving the initial value problem

$$y'' + y = \sin t, \qquad y(0) = 2, y'(0) = 1.$$

The first steps are routine:

$$\begin{aligned}\mathcal{L}\{y'' + y\} &= s^2F(s) - y'(0) - sy(0) + F(s) = (s^2+1)F(s) - 2s - 1,\\ \mathcal{L}\{\sin t\} &= \frac{1}{s^2+1}.\end{aligned}$$

Thus,

$$F(s) = \frac{1}{(1+s^2)^2} + \frac{1+2s}{1+s^2}.$$

Now, in the expression of $F(s)$, we know what the inverse Laplace transform of the second term is; however, what is $\mathcal{L}^{-1}\{(1+s^2)^{-2}\}$? This is not obvious from the list of Laplace transforms, nor from the shifting properties. However, one may realize that this has something to do with the derivative of the functions $s(1+s^2)^{-1}$ and $(1+s^2)^{-1}$, whose inverse Laplace transforms are well known. Also, we might need to used the derivative property:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s).$$

In fact,

$$\begin{aligned}
\frac{d}{ds}\frac{s}{1+s^2} &= \frac{1}{1+s^2} - \frac{2s^2}{(1+s^2)^2} = \frac{1-s^2}{(1+s^2)^2} \\
&= \frac{-1-s^2+2}{(1+s^2)^2} \\
&= \frac{2}{(1+s^2)^2} - \frac{1}{1+s^2}.
\end{aligned}$$

Therefore,

$$\frac{1}{(1+s^2)^2} = \frac{1}{2}\frac{d}{ds}\frac{s}{1+s^2} + \frac{1}{2}\frac{1}{1+s^2}.$$

Hence, we can use the linearity and one of the derivative properties and get

$$\mathcal{L}^{-1}\{(1+s^2)^{-2}\} = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t.$$

Conclusion:

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{(1+s^2)^{-2} + (1+2s)(1+s^2)^{-1}\} \\
&= -\frac{1}{2}t\cos t + \frac{3}{2}\sin t + 2\cos t.
\end{aligned}$$