Math 3001-01 Spring 2020
Practice Exam I

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the CU Community Standard.

Name: Solutions　　　　Signature: ____________

Instructions:

- Notes, books, calculators or computers are not allowed in this exam.
- Make sure you show the work that leads to your answer to receive full credit. If you are using a theorem or a fact to draw some conclusions, quote the result.
- This test contains **4 questions**. You have **50 minutes** to answer all the questions.

Good Luck !

Question	Score
1	/ 25
2	/ 25
3	/ 25
4	/ 25
Total	**/100**

1. This question is concerned with the "Archimedean Property".

(1) Write down the content of the **Archimedean Property** of real numbers.

For any real number $y > 0$, there exists a natural number $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$.

(2) Use the Archimedean Property to prove that

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$

Suppose that $y \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$; we have $y > 0$.

By the Archimedean Property of real numbers, $\exists\, m \in \mathbb{N}$ such that $0 < \frac{1}{m} < y$.

Thus, $y \notin (0, \frac{1}{m})$. This is a contradiction.

$\therefore \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi.$

(3) Another consequence of the Archimedean Property is that both $\mathbb{Q}$ and $\mathbb{I}$ are (fill in the box below with a suitable word)

$\boxed{\text{dense}}$

in $\mathbb{R}$.

2. This question is concerned with "cardinality".

(1) Let A, B be two sets. What does it mean by saying that A and B have the **same cardinality**? (Write down the definition.)

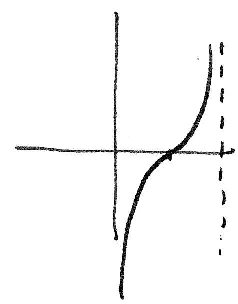
A, B have the <u>same cardinality</u> if there exists a <u>1-1 and onto map</u> $f: A \to B$.

(2) Use the definition to prove that the closed interval $(0, 1)$ and $\mathbb{R}$ have the same cardinality. (You need to provide a single function that meets the requirements in the definition.)

Consider the function

$$f: (0,1) \to \mathbb{R}$$

defined by $$f(x) = \frac{x - \frac{1}{2}}{x(1-x)}$$



This function is 1-1 and onto.

Indeed, for $y=0$, $f(x)=0 \Rightarrow x=0$

and for $y \neq 0$, $f(x) = y \Rightarrow x = \dfrac{(y-1) \pm \sqrt{1+y^2}}{2y}$

This part you don't have to worry about.

the only valid x is

$$x = \frac{y-1+\sqrt{1+y^2}}{2y}$$

and it belongs to $(0,1)$ for all $y \neq 0$.

3. This question is concerned with the "Monotone Convergence Theorem".

(1) State the **Monotone Convergence Theorem** of sequences.

$(a_n)_{n=1}^{\infty}$ bounded and monotone $\Rightarrow$ $(a_n)_{n=1}^{\infty}$ converges.

(2) Define a sequence (a_n) by

$$a_1 = \sqrt{2}, \quad a_{k+1} = \sqrt{2 + a_k} \quad (k = 1, 2, \ldots).$$

Use mathematical induction to show that (a_n) is increasing.

Step 1: $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}} > a_1$

Step 2: if $0 < a_k \leq a_{k+1}$, then $a_{k+1} = \sqrt{2+a_k}$
$\leq \sqrt{2+a_{k+1}}$
$= a_{k+2}$.

$\therefore$ By induction, $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

(3) Is the sequence in part (2) convergent? If so, what is the limit? Justify your answer.

$a_1 \leq 2$, now if $a_k \leq 2$ then $a_{k+1} = \sqrt{2+a_k} \leq \sqrt{2+2}$
$= 2$

Therefore, $a_k \leq 2$ for all $k \in \mathbb{N}$.

(a_n) is bounded and monotone.

By MCT, $\lim_{n\to\infty} a_n$ exists. Suppose $\lim_{n\to\infty} a_n = a$.

We have $a = \sqrt{2+a}$ $\Rightarrow$ $a^2 - a - 2 = 0$.

$\Rightarrow$ $\boxed{a = 2}$.

4. For each of the following, either give an example that satisfies the condition(s), or explain why an example does not exist.

(1) A sequence $(a_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} a_n = 0$ and $\sup\{a_n\}_{n=1}^{\infty} = 1$. (Note: $\{a_n\}_{n=1}^{\infty}$ is the **set** formed by all distinct numbers that occur in (a_n).)

Example: $(a_n) = \left(\frac{1}{n}\right)$.

(2) A convergent sequence (a_n) in which both 0 and 1 occur infinitely many times.

Impossible. We'd have two subsequences converging to distinct limits.

(3) A sequence of **closed intervals** I_n satisfying $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$ and

$$\bigcap_{n=1}^{\infty} I_n = \{2020, 2019\}.$$

Impossible.

if $2019 \in I_n$ for all $n \in \mathbb{N}$

and $2020 \in I_n$ for all $n \in \mathbb{N}$.

then $[2019, 2020] \subset I_n, \quad \forall n \in \mathbb{N}$

thus $[2019, 2020] \subseteq \bigcap_{n=1}^{\infty} I_n$.