Math 3001-01 Spring 2020
Exam I

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the CU Community Standard.

Name: Solutions Signature: ____________________

Instructions:

- Notes, books, calculators or computers are not allowed in this exam.
- Make sure you show the work that leads to your answer to receive full credit. If you are using a theorem or a fact to draw some conclusions, quote the result.
- This test contains **4 questions**. You have **50 minutes** to answer all the questions.

Good Luck !

Question	Score
1	/ 25
2	/ 25
3	/ 25
4	/ 25
Total	/100

1. This question is about the **supremum** of subsets of $\mathbb{R}$.

(1) Let $A \subset \mathbb{R}$ be a nonempty subset. As long as A is (write down a suitable word in each of the two boxes below)

$\boxed{\text{bounded}}$ $\boxed{\text{above}}$,

there exists a number $s \in \mathbb{R}$ that satisfies the following two conditions:

i. $\forall\ a \in A,\quad a \leq s$ (s is an upper bound of A)

ii. $\forall\ s'$ upper bound of A, we have $s \leq s'$.

Such an s is called the **supremum** of A; and the fact described above is called the

Axiom of $\boxed{\text{Completeness}}$.

(2) Determine the **suprema** of the following subsets of $\mathbb{R}$, if they exist.
(You do *not* need to prove your conclusions.)

i. $\left(0, \sqrt{3}\right) \cap \mathbb{Q}$ $\sqrt{3}$

ii. $\displaystyle\bigcap_{n=1}^{\infty} \left(-\infty, \frac{2}{n}\right)$ 0

iii. $\displaystyle\bigcup_{n=1}^{\infty} (n, n+1)$ doesn't exist

2. This question is concerned with "cardinality".

(1) Let A be a set. What does it mean by saying that A is **countable**? (Write down the definition.)

A has the same cardinality as $\mathbb{N}$.

or ($\exists f: \mathbb{N} \to A$ that is 1-1 and onto.)

(2) Cite a known property of countable sets to prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countable.

$$\mathbb{N} \times \mathbb{N} = \bigcup_{n=1}^{\infty} \{n\} \times \mathbb{N}$$

This is a countable union of countable sets.

hence it is countable.

3. This question is concerned with the "Monotone Convergence Theorem".

(1) State the **Monotone Convergence Theorem** of sequences.

$$\left.\begin{array}{l} (a_n) \text{ sequence} \\ \text{bounded} \\ \text{monotone} \end{array}\right\} \Rightarrow (a_n) \text{ converges.}$$

(2) Define a sequence (a_n) by

$$a_1 = 1, \quad a_{k+1} = \frac{1}{5 - a_k} \quad (k = 1, 2, \ldots).$$

Use mathematical induction to show that (a_n) is decreasing.

$a_1 = 1, \quad a_2 = \frac{1}{4} \qquad \therefore \qquad a_1 \geq a_2$

now assume that $\quad a_k \geq a_{k+1}$ for some $k \in \mathbb{N}$.

then $\quad a_{k+1} = \dfrac{1}{5 - a_k} \geq \dfrac{1}{5 - a_{k+1}} = a_{k+2}$

By induction, $\quad a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$.

(3) Does the sequence in part (2) converge? If so, what is the limit? Justify your answer.

Yes. We have $a_k \leq 1$ and therefore by $a_{k+1} = \dfrac{1}{5 - a_k}$,

$a_k \geq 0$ for all k.

So (a_n) is decreasing and bounded below.

By MCT, (a_n) converges.

The limit can be found by solving: $\quad a = \dfrac{1}{5-a}$

$\Rightarrow \quad a^2 - 5a + 1 = 0$

$a = \dfrac{5 \pm \sqrt{21}}{2}$

$\boxed{\text{valid "}a\text{": } a = \dfrac{5 - \sqrt{21}}{2}}$

4. For each of the following, either give an example that satisfies the condition(s), or explain why an example does not exist.

(1) A divergent sequence (a_n) all whose convergent subsequences share the same limit.

Example: $(1, 2, 1, 3, 1, 4, 1, 5, \dots)$

(2) A convergent sequence (a_n) that is **not** monotone.

Example: $\left(\frac{(-1)^n}{n}\right)$.

(3) A sequence (a_n) that satisfies: $0 < a_n < 2020$ for all $n \in \mathbb{N}$, and all subsequences of (a_n) diverge.

Impossible, by Bolzano-Weierstraß.