PROBLEM SET 4

Due: Wed., Apr. 5

1. Suppose that a function $f(x)$ has its Taylor series expansion about $x = 0$ being

$$\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^2 \cdot 2^k} x^k.$$

i. What is the value of $f^{(5)}(0)$.

ii. What is the *interval of convergence* of the Taylor series above?

iii. Approximating the value of $f(1)$ with $P_5(1)$, where $P_5(x)$ is the fifth-degree Taylor polynomail of $f(x)$ about $x = 0$, what is an upper bound for the error incurred by this approximation?

2. Estimate

$$\int_0^1 \frac{\sin x}{x} dx$$

using the fifth-degree Taylor polynomial of $\int \frac{\sin x}{x} dx$. What is an upper bound for the error incurred by this approximation?

3. Given a polynomial function, say, $p(x) = 1 + x^2$, its Taylor series about $x = 0$ takes the same form as the function itself. To find its Taylor series about $x = 1$, we could write x as $(x-1)+1$ and expand $1+((x-1)+1)^2$, obtaining $2+2(x-1)+(x-1)^2$. The coefficients 2, 2 and 1 are the values of $p(1)$, $p'(1)$ and $\frac{p''(1)}{2}$, respectively.

Now, use the same idea to find (without taking derivatives) the Taylor series of the polynomial function $q(x) = 1 + 2x - x^3$ about $x = -2$. How do the coefficients in the Taylor series you found relate to the function $q(x)$ and its derivatives?

4. Let $A(x)$ denote the power series $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$, noting that the radius of convergence is equal to 1. Write down the power series corresponding to $\frac{d^2}{dx^2}(x \cdot A(x))$. Which familiar function does the power series you wrote down correspond to (as its Taylor series about $x = 0$)? Then, find the closed-form expression of $A(x)$ as a function defined on $(-1, 1)$.

5. The following statements are *false*. Find out why. (Note: For **iii**, give a counter-example to show that the statement is false.)

i. The Taylor series for e^x about $x = 1$ is

$$1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + ...$$

ii. The Taylor series for $\frac{1}{(1-x)^2}$ about $x = 0$ can be computed by

$$\left(1 + x + x^2 + x^3 + ...\right)^2 = 1 + x^2 + x^4 + x^6 + ...$$

iii. As long as $f(x)$ is defined at $x = a$, its Taylor series (wherever it is centered) would converge at $x = a$.

6†. Recall from a previous chapter the Simpson's approximation for a definite integral (see p. 391 of the textbook):

$$\text{SIMP}(n) = \frac{2\text{MID}(n) + \text{TRAP}(n)}{3}.$$

The goal of this problem is to help you see why the $2:1$ ratio of MID and TRAP makes sense.

Let $f(x)$ be a differentiable function, whose Taylor series about $x = 0$ has its interval of convergence containing $[-h, h]$ $(h > 0)$. The goal is to approximate the integral

$$\int_{-h}^{h} f(x)dx.$$

i. Write down the Taylor series of $f(x)$ about $x = 0$; then write down the Taylor series of $\int f(x)dx$ about $x = 0$. For the latter, you may write the undetermined constant term as C.

ii. For this part, use the Taylor series you wrote down in the previous part to express each of the following quantities as an infinite series in terms of $f^{(k)}(0)$ $(k = 0, 1, 2, ...)$ and h:

$$\text{MID}(1); \qquad f(h),\ f(-h),\ \text{TRAP}(1); \qquad \int_{-h}^{h} f(x)dx.$$

iii. Use the expressions you found in the previous part to show that in the infinite series expression of

$$\int_{-h}^{h} f(x)dx - \frac{2\text{MID}(1) + \text{TRAP}(1)}{3},$$

the coefficients of h^k are zero for $k = 0, 1, 2, 3, 4$.