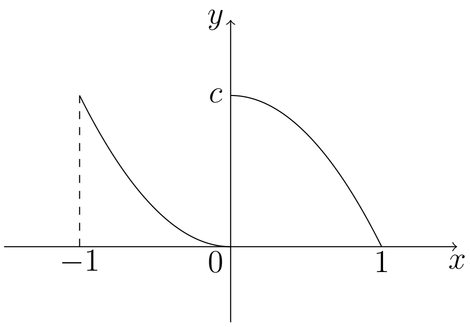
PROBLEM SET 3 SOLUTION

Part I

1. **i**. The graph of $f_x(x)$ looks like **ii**. Though it is possible to tell immediately from the



symmetry in the graph that $c = 1$, we could still use calculate to obtain the value; namely, by calculating

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f_c(x) &= \int_{-1}^{0} cx^2 dx + \int_0^1 (c - cx^2) dx \\ &= \frac{cx^3}{3}\Big|_{-1}^{0} + \left(cx - \frac{c}{3}x^3\right)\Big|_0^1 \\ &= \frac{c}{3} + c - \frac{c}{3} \\ &= c. \end{aligned}$$

Hence,

$$c = 1.$$

iii. If $x < -1$, then

$$F(x) = \int_{-\infty}^{x} 0 dt = 0;$$

if $-1 \le x < 0$, then

$$F(x) = \int_{-\infty}^{x} f_1(t) dt = \int_{-1}^{x} t^2 dt = \frac{t^3}{3}\Big|_{-1}^{x} = \frac{1}{3} + \frac{x^3}{3};$$

if $0 \le x < 1$, then

$$F(x) = \int_{-\infty}^{x} f_1(t) dt = \int_{-1}^{0} t^2 dt + \int_0^x (1 - t^2) dt = \frac{1}{3} + x - \frac{x^3}{3};$$

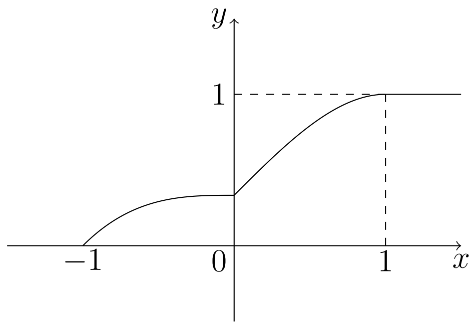
if $x \ge 1$, then

$$F(x) = 1.$$

In summary,

$$F(x) = \begin{cases} 0, & t < -1, \\ \frac{1}{3} + \frac{x^3}{3}, & -1 \le x < 0, \\ \frac{1}{3} + x - \frac{x^3}{3}, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

The graph of $F(x)$ looks like:



2. **i**. The integral $\int_{-2}^{3} \frac{1}{x^3}\, dx$ is improper, it can be written as

$$\int_{-2}^{3} \frac{1}{x^3}\, dx = \int_{-2}^{0} \frac{1}{x^3}\, dx + \int_{0}^{3} \frac{1}{x^3}\, dx,$$

where neither of the integrals on the right hand side are convergent. As a result, the original integral diverges. (Caution: One should not apply the fundamental theorem of calculus here, as the original integrand is undefined at $x = 0$.)

ii. Using partial fractions, we let

$$\frac{3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A + Cx + (A+B)x^2}{x(x^2+1)},$$

and it can be solved that

$$A = 3, \quad B = -3, \quad C = 0.$$

Hence,

$$\begin{aligned}\int_1^2 \frac{3}{x(x^2+1)}dx &= \int_1^2 \frac{3}{x}dx - \int_1^2 \frac{3x}{1+x^2}dx \\ &= 3\ln x\Big|_1^2 - \frac{3}{2}\ln(1+x^2)\Big|_1^2 \\ &= 3\ln 2 - \frac{3}{2}(\ln 5 - \ln 2) \\ &= \frac{9}{2}\ln 2 - \frac{3}{2}\ln 5.\end{aligned}$$

iii. Integration by parts. Letting $u = x$, $v' = \cos 2x$; then $u' = 1$, $v = \frac{1}{2}\sin 2x$.

$$\begin{aligned}\int_0^1 x\cos 2x\, dx &= \frac{1}{2}x\sin 2x\Big|_0^1 - \int_0^1 \frac{1}{2}\sin 2x dx \\ &= \frac{1}{2}\sin 2 + \frac{1}{4}\cos 2x\Big|_0^1 \\ &= \frac{1}{2}\sin 2 + \frac{1}{4}(\cos 2 - 1).\end{aligned}$$

iv. With the u-substitution: $u = \arctan x$, thus $du = \frac{1}{\arctan x}dx$, the integral becomes

$$\int_{\pi/4}^{\pi/2} \frac{1}{u}du = \ln u\Big|_{\pi/4}^{\pi/2} = \ln 2.$$

v. Since the integrand is an odd function, and the integration is over a symmetric interval, the integral equals to zero.

vi. Again, the integrand is odd, and the interval is symmetric; however, one has to treat this as an improper integral:

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \lim_{b \to -\infty} \int_b^0 xe^{-x^2} dx + \lim_{b \to \infty} \int_0^b xe^{-x^2} dx.$$

Note that xe^{-x^2} is continuous and,

$$\int xe^{-x^2} dx = -\frac{1}{2} e^{-x^2},$$

we have

$$\lim_{b \to \infty} \int_0^b xe^{-x^2} dx = \frac{1}{2}, \qquad \lim_{b \to -\infty} \int_b^0 xe^{-x^2} dx = \frac{-1}{2}.$$

Thus the original integral converges and equals to zero.

vii. Let $u = 1 + x^2$, then $du = 2xdx$. The integral becomes

$$\frac{1}{2} \int_1^2 \sqrt{u}\; du = \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^2 = \frac{1}{3}(2\sqrt{2} - 1).$$

viii. Partial fraction gives

$$\frac{1}{x^4 - 1} = \frac{1}{2(x^2 - 1)} - \frac{1}{2(x^2 + 1)} = \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} - \frac{1}{2(x^2 + 1)}.$$

Thus the integral equals to

$$\left(\frac{1}{4} \ln |x - 1| - \frac{1}{4} \ln |x + 1| - \frac{1}{2} \arctan(x) \right) \Big|_2^3 = \frac{1}{4}(\ln 3 - ln 2) - \frac{1}{2}(\arctan(3) - \arctan(2)).$$

3. According to the length formula for parametrized curves, we have

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\; dt \\
&= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t\; dt} \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t}\; dt \\
&= \int_0^{2\pi} \sqrt{4 \sin^2(t/2)}\; dt \\
&= 2 \int_0^{2\pi} \sin \frac{t}{2}\; dt \\
&= -4 \cos \frac{t}{2} \Big|_0^{2\pi} \\
&= 8.
\end{aligned}$$

4. By the length formula, we have

$$\begin{aligned} L &= \int_0^1 \sqrt{1+(f'(x))^2}\ dx \\ &= \int_0^1 \sqrt{1+\frac{9}{4}x}\ dx \\ (u = 1+\tfrac{9}{4}x) &= \frac{4}{9}\int_1^{13/4} \sqrt{u}\ du \\ &= \frac{8}{27}u^{3/2}\Big|_1^{13/4} \\ &= \frac{13\sqrt{13}}{27} - \frac{8}{27}. \end{aligned}$$

5. By the volume formula, we have

$$\begin{aligned} V &= \int_0^\pi \pi \sin^2 x dx \\ &= \int_0^\pi \frac{\pi}{2}(1-\cos 2x)dx \\ &= \frac{\pi^2}{2} - \frac{\pi}{4}\sin 2x\Big|_0^\pi \\ &= \frac{\pi^2}{2}. \end{aligned}$$

6. No. A cumulative distribution function must tend to 1 as x tends to infinity, which means its integral over $(-\infty, \infty)$ must be infinity, rather than 1, which is the case for a probability density function.

Part II

i.

$$\begin{aligned} \text{PV} &= \int_{2017}^L e^{-r(t-2017)}p(t)dt, \\ \text{FV} &= \int_{2017}^L e^{r(L-t)}p(t)dt, \end{aligned}$$

ii. It makes sense because, in the case when the interest rate compounds continuously at a constant rate r, the future value must be the present value multiplied by e^{rT}, where T is the *difference* between the future and present times.

iii. The present value for all the income starting from 2017 is

$$\int_{2017}^\infty 2(t-2017)e^{-r(t-2017)}dt.$$

With the substitution $u = t - 2017$, this integral becomes

$$\int_0^\infty 2ue^{-ru}du,$$

which, by integration by parts, equals to $\frac{2}{r^2}$. Since $r = 0.05$, we obtain that the present value of all the income is 800 thousand dollars, hence choose selling the mart now for better profit.