PROBLEM SET 3

Due: Mon., Mar. 6

Part I

1. You're given a function

$$f_c(x) = \begin{cases} 0, & x < -1, \\ cx^2, & -1 \le x < 0, \\ c - cx^2, & 0 \le x < 1, \\ 0, & x \ge 1. \end{cases} \qquad (c > 0)$$

i. Sketch the graph of $f_c(x)$.
ii. For which value of c does $f_c(x)$ represent a *probability density function*?
iii. Find the *cumulative distribution function* $F(x)$ associated to the density function $f_c(x)$ you determined in part ii. Plot the graph of $F(x)$.

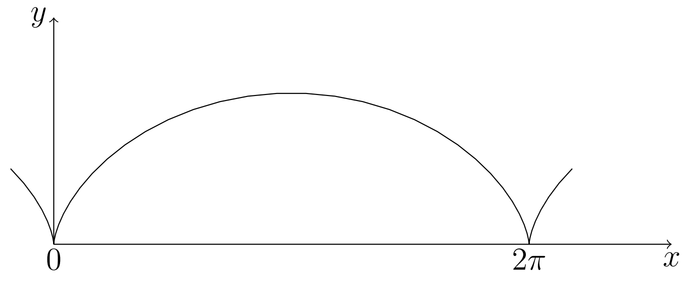
2. Evaluate the following integrals.

i. $\displaystyle\int_{-2}^{3} \frac{1}{x^3}\, dx$; **ii**. $\displaystyle\int_{1}^{2} \frac{3}{x(x^2+1)} dx$; **iii**. $\displaystyle\int_{0}^{1} x \cos 2x \, dx$; **iv**. $\displaystyle\int_{1}^{\infty} \frac{1}{(x^2+1)\arctan x}\, dx$;
v. $\displaystyle\int_{-1}^{1} \sin^{2017} x \, dx$; **vi**. $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\, dx$; **vii**. $\displaystyle\int_{0}^{1} x\sqrt{1+x^2}\, dx$; **viii**. $\displaystyle\int_{2}^{3} \frac{1}{x^4-1} dx$.

3. When you roll a circle (assumed to have radius 1) along a straight line without slip, the point on the circle which initially touches the line will trace out a curve, known as a *cycloid*. Let t denote the angle being rotated from the beginning, then such a cycloid has the parametrization

$$(x(t), y(t)) = (t - \sin t,\ 1 - \cos t), \qquad t \ge 0.$$

Compute the length of one arch of this cycloid, namely, between $0 \le t \le 2\pi$.



4. Compute the length of the graph of $f(x) = x^{3/2}$ within $0 \le x \le 1$.

5. Consider the graph of $y = \sin x$ along the inverval $0 \le x \le \pi$. Rotating it about the x-axis produces a shape that looks like an American football. Find the volume of this shape.

6. Let $f(x)$ be a function defined on $(-\infty, \infty)$ and $f(x) \ge 0$. Is it possible for $f(x)$ to be both a probability density function and a cumulative distribution function? Explain.

Part II

This part is concerned with the topic of present/future values.

Supposing that during $2017 \leq t \leq L$ (in year), the continuous income stream (from your mart) is predicted to be at rate of $p(t)$ (thousands dollars per year), the (yearly) interest rate being a constant r (interest compounded continuously).
i. Write down the formulas for the present value (PV) (at $t = 2017$) and the future value (FV) (at $t = L$) for the entire income during this time.
ii. Observe that, for the PV and FV you found in part i, you have $\text{FV} = e^{r(L-2017)} \cdot \text{PV}$. Why does this make sense?
iii. Now suppose that $p(t)$ and r are given:

$$p(t) = 2(t - 2017), \qquad r = 0.05,$$

and let L be large enough; you are deciding in between running the mart from $t = 2017$ to $t = L$ and selling it now (at $t = 2017$) for \$850,000. Which would you choose for better profit?