PROBLEM SET 2 SOLUTION

Due: Fri., Feb.24

Part I

This part contains two problems. The first finishes an integral which is not completed in our textbook, whose solution involves a combination of integration by substitution, partial fractions, and integration by parts; but if you get to problem "$1\frac{1}{2}$", a simpler and alternative way to integrate $\int \sqrt{1+x^2}\, dx$ is available; the second, using integration by parts only and thus obtaining recurrence formulas, we arrive at an infinite product that approximates π. These questions are somewhat challenging in themselves, but, hopefully, become managable as they are broken down into pieces, and the goal is for you to see how far you could go with what you've learned already.

I.1. In class we used the substitution $x = \dfrac{1}{\tan\theta}$ $(\theta \in (-\pi/2, \pi/2))$ to transform the integral

$$\int_0^1 \sqrt{1+x^2}\, dx$$

to

$$\int_0^{\pi/4} \frac{1}{\cos^3\theta} d\theta,$$

(see also, Hughes-Hallett pp. 382-383), whose value still remains to be found. In fact, if one uses $\cos^2\theta + \sin^2\theta = 1$, the latter integral can be expressed as

$$\int_0^{\pi/4} \frac{1}{\cos\theta} d\theta + \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^3\theta} d\theta.$$

For the first member in the summation, one could again use the trigonometry identity mentioned to rewrite it as

$$\int_0^{\pi/4} \frac{1}{\cos\theta} d\theta = \int_0^{\pi/4} \cos\theta d\theta + \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta} d\theta.$$

As a result, we have

$$\int_0^{\pi/4} \frac{1}{\cos^3\theta} d\theta = \underbrace{\int_0^{\pi/4} \cos\theta d\theta}_{P} + \underbrace{\int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta} d\theta}_{Q} + \underbrace{\int_0^{\pi/4} \frac{\sin^2\theta}{\cos^3\theta} d\theta}_{R}.$$

For the expression on the right hand side of the previous equality, the first member is obvious to integrate:

$$P = \int_0^{\pi/4} \cos\theta d\theta = \sin\theta \Big|_0^{\pi/4} = \frac{\sqrt{2}}{2}.$$

For the second member Q, if we make the substitution $u = \sin\theta$ (noting that this substitution makes sense as x and θ admit a one-to-one correspondence, since θ belongs to $(-\pi/2, \pi/2)$), then we have

$$du = \cos\theta d\theta.$$

i.Show that, under this substitution, and by using the previously mentioned trigonometry identity, the integral $Q = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta} d\theta$ is transformed to

$$Q = \int_0^{\sqrt{2}/2} \frac{u^2}{1-u^2} du.$$

As $u = \sin\theta$, we have $du = \cos\theta d\theta$, hence

$$Q = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos\theta d\theta = \int_0^{\sqrt{2}/2} \frac{u^2}{1-u^2} du.$$

ii.Evaluate the integral you found in part **i** by expanding its integrand using partial fractions:

$$\frac{u^2}{1-u^2} = A + \frac{B}{1+u} + \frac{C}{1-u}.$$

(Note: When you integrate $\int (1-u)^{-1} du$, be careful with the negative sign!)

Noting that

$$\frac{u^2}{1-u^2} = -1 + \frac{1}{1-u^2},$$

one only needs to apply partial fractions to $\frac{1}{1-u^2}$, say,

$$\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u},$$

which requires

$$A(1+u) + B(1-u) \equiv 1.$$

By matching the coefficients of different powers of u on both sides of the previous equation, we have $A + B = 1$ and $A - B = 0$, which leads to

$$A = B = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} Q = \int_0^{\sqrt{2}/2} \frac{u^2}{1-u^2} &= \int_0^{\sqrt{2}/2} \left(-1 + \frac{1}{2(1-u)} + \frac{1}{2(1+u)}\right) \\ &= \left(-u - \frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u|\right)\Big|_0^{\sqrt{2}/2} \\ &= -\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right). \end{aligned}$$

iii.Use the method of integration by parts, by taking $u = \sin\theta$, $v = \dfrac{1}{2\cos^2\theta}$, to show that

$$\begin{aligned} R = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^3\theta} d\theta &= \sin\theta \cdot \frac{1}{2\cos^2\theta}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos\theta} d\theta \\ &= \sin\theta \cdot \frac{1}{2\cos^2\theta}\Big|_0^{\pi/4} - \frac{1}{2}(P+Q). \end{aligned}$$

With the given choice of u, v, we have $u' = \cos\theta$, $v' = \dfrac{\sin\theta}{\cos^3\theta}$, hence $uv' = \dfrac{\sin^2\theta}{\cos^3\theta}$ and $u'v = \dfrac{1}{2\cos\theta}$. Now, by integration by parts,

$$R = \int_0^{\pi/4} uv' d\theta = \underbrace{\sin\theta \cdot \frac{1}{2\cos^2\theta}}_{uv}\Big|_0^{\pi/4} - \int_0^{\pi/4} \underbrace{\frac{1}{2\cos\theta}}_{u'v} d\theta.$$

The desired formula then follows immediately by noting that $\int_0^{\pi/4} \frac{1}{\cos\theta} d\theta$ is equal to $P + Q$.

iv. Finish up by finding the original integral, which is now, symbolically, $P + Q + R$. Just for you to check your answer, it should be consistent with the formula for the indefinite integral

$$\int \frac{1}{\cos^3\theta} d\theta = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) + c,$$

where c is an arbitrary constant.

By the previous part, we have

$$R = \frac{\sqrt{2}}{2} - \frac{1}{2}(P + Q),$$

hence, by adding $P + Q$ on both sides,

$$P + Q + R = \frac{\sqrt{2}}{2} + \frac{1}{2}(P + Q).$$

By previous calculation, we have

$$P + Q = \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

Thus,

$$P + Q + R = \frac{\sqrt{2}}{2} + \frac{1}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

I.1$\frac{1}{2}$. There is an alternative way to find the integral $\int \sqrt{1+x^2}\, dx$, that is, by substituting using the hyperbolic trig. functions. Recall the definition

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2},$$

as well as several properties as one could easily verify:

$$\cosh^2 t - \sinh^2 t = 1,$$

$$\frac{d}{dt}\cosh t = \sinh t, \quad \frac{d}{dt}\sinh t = \cosh t.$$

The property $\cosh^2 t - \sinh^2 t = 1$ motivates the term "hyperbolic", as $(\cosh t, \sinh t)$ always lie on a branch of the hyperbola $u^2 - v^2 = 1$ in the u-v plane.
Now apply the substitution

$$x = \sinh t$$

to the integral $\int \sqrt{1+x^2}\, dx$ to show that it equals to

$$\frac{1}{4}\sinh 2t + \frac{1}{2}t + c,$$

where $t = \sinh^{-1}(x)$ (the "-1" indicates the inverse of a function, instead of the power of $\sinh x$) and c is an arbitrary constant.

By making the substitution $x = \sinh t$, we have $dx = \cosh t\ dt$, and $\sqrt{1+x^2} = \sqrt{1+\sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$. Hence,

$$\int \sqrt{1+x^2}dx = \int \cosh t(\cosh t\ dt) = \int \cosh^2 t dt.$$

Since

$$\cosh^2 t = \frac{e^{2t}+e^{-2t}+2}{4} = \frac{1}{2}\cosh 2t + \frac{1}{2},$$

we have

$$\int \cosh^2 t dt = \int \frac{1}{2}\cosh 2t + \frac{1}{2} dt = \frac{1}{4}\sinh 2t + \frac{1}{2}t + c,$$

as desired.

I.2. An infinite product that approximates π can be found by studying the integrals

$$I_m := \int_0^{\pi/2} \sin^m x\ dx, \qquad m = 0, 1, 2, ...$$

using integration by parts.

i. For $m = 0, 1$ each, evaluate the definite integral $I_m = \int_0^{\pi/2} \sin^m x\ dx$.

For $m = 0$,

$$I_0 = \int_0^{\pi/2} 1\ dx = \frac{\pi}{2};$$

for $m = 1$,

$$I_1 = \int_0^{\pi/2} \sin x dx = -\cos x\Big|_0^{\pi/2} = 1.$$

ii. If $m \geq 2$, one could write the integral I_m as

$$I_m = \int_0^{\pi/2} (\sin x)^{m-1}(\sin x) dx.$$

Use integration by parts once to show that

$$I_m = \int_0^{\pi/2} (m-1)(\sin x)^{m-2}(\cos x)^2 dx;$$

then, by writing $\cos^2 x$ as $1 - \sin^2 x$, show that

$$I_m = \frac{m-1}{m} I_{m-2}.$$

Taking $u = \sin^{m-1} x$, $v = -\cos x$, we have

$$\begin{aligned} uv' &= \sin^{m-1} x \sin x = \sin^m x, \\ uv &= -\sin^{m-1} x \cos x, \\ u'v &= -(m-1)\sin^{m-2} x \cos^2 x. \end{aligned}$$

Using the integration by parts formula, we have

$$I_m = -\sin^{m-1} x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} (m-1)\sin^{m-2} x \cos^2 x \; dx.$$

The first term on the right hand side vanishes, because $\cos(\pi/2) = 0$ and because $m \geq 2$, $\sin 0 = 0$. The second term can be rewritten as

$$\begin{aligned} &\int_0^{\pi/2} (m-1)\sin^{m-2} x(1-\sin^2 x) \; dx \\ &= (m-1)\int_0^{\pi/2} \sin^{m-2} x \; dx - (m-1)\int_0^{\pi/2} \sin^m x \; dx \\ &= (m-1)I_{m-2} - (m-1)I_m, \end{aligned}$$

which, by the previous argument, also equals to I_m. Hence, one easily solves from $I_m = (m-1)I_{m-2} - (m-1)I_m$ and get

$$I_m = \frac{m-1}{m} I_{m-2}.$$

iii.(Reading) Let n denote sufficiently large positive integers. According to parts **i** and **ii**, you should have,

$$\begin{aligned} I_{2n+1} &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \ldots \cdot \frac{2}{3} \cdot 1, \\ I_{2n} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2}. \end{aligned}$$

In particular, since $\sin x \leq 1$ for $x \in [0, \pi/2]$, we have

$$(\sin x)^{2n+1} \leq (\sin x)^{2n} \leq (\sin x)^{2n-1},$$

and thus

$$I_{2n+1} \leq I_{2n} \leq I_{2n-1}.$$

Dividing these inequalities by I_{2n+1} and using part **ii**, we have

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n+1}{2n}.$$

Letting $n \to \infty$, one is forced to have

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

On the other hand, I_{2n}/I_{2n+1} is equal to

$$\frac{\pi}{2}\left(\frac{1\cdot 3}{2^2} \cdot \frac{3\cdot 5}{4^2} \cdot \ldots \cdot \frac{(2n-1)(2n+1)}{(2n)^2}\right).$$

Combining facts above, we have

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^2}{1\cdot 3} \cdot \frac{4^2}{3\cdot 5} \cdot \ldots \cdot \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Part II

II.1. We learned in class that the integral $\int_1^\infty \frac{1}{x^p}$ converges if and only if $p > 1$. Now use a comparison argument, *without evaluating the integrals*, to determine the convergence of the following improper integrals:

$$(a) \int_1^\infty \frac{1}{\sqrt{x+x^2}}, \qquad (b) \int_1^\infty \frac{3}{(x^5+4x^3+2x+1)^{1/3}}.$$

(a) Noting that $\frac{1}{\sqrt{x+x^2}} \geq \frac{1}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{x}$ for $x \geq 1$, and that $\int_1^\infty \frac{1}{x}$ diverges, we see that this given improper integral is divergent. (Note: You could also invoke the "integral test" from the series chapter, but applied in a reversed way.)

(b) When $x \geq 1$, we have $\frac{3}{(x^5+4x^3+2x+1)^{1/3}} \leq \frac{3}{(x^5)^{1/3}} = 3 \cdot \frac{1}{x^{5/3}}$. Since the integral $\int_1^\infty \frac{1}{x^{5/3}}$ converges, the given improper integral also converges; where we have used the fact that the integrand is always positive.

II.2. Supposing that $\int_0^1 f'(\sin x)\cos x\, dx = 10$, and $f(0) = 1$, what's the value of $f(\sin 1)$?

Letting $u = \sin x$, we have $du = \cos x dx$, then the given integral becomes

$$\int_{u(0)}^{u(1)} f'(u)du = f(u)\Big|_{u=u(0)}^{u=u(1)} = f(\sin(1)) - f(\sin(0)) = f(\sin 1) - 1.$$

Clearly, $f(\sin 1) = 11$.

II.3. Let $f(x)$ be a continuous, increasing, concaving-down function defined on the interval $[0, 1]$. Put the following quantities in increasing order:

$$\int_0^1 f(x)dx, \qquad \text{LEFT}(n), \qquad \text{RIGHT}(n), \qquad \text{MID}(n), \qquad \text{TRAP}(n).$$

$$\text{LEFT}(n) \leq \text{TRAP}(n) \leq \int_0^1 f(x)dx \leq \text{MID}(n) \leq \text{RIGHT}(n).$$

II.4. Supposing that for a certain function $f(x)$ defined on $[1, 2]$, you only know that it satisfies $\int_1^2 f(x)dx = 8$, is it possible to find each of the following integrals? What if you know in addition that $f(2) = 2$, does your answer change?

$$(a) \int_3^9 x\, f(2x^2+1)dx, \qquad (b) \int_1^2 (x-1)f'(x)dx.$$

(a) Making the substitution $u = 2x^2 + 1$ transforms the integral into

$$\frac{1}{4}\int_{19}^{163} f(u)du,$$

whose value is impossible to be determined with what's given.
(b) Using integration by parts, and letting $u = x - 1$, $v = f(x)$, we have $uv = (x-1)f(x)$, $u'v = f(x)$, and thus

$$\int_1^2 (x-1)f'(x)dx = (x-1)f(x)\Big|_1^2 - \int_1^2 f(x)dx = f(2) - 8,$$

which equals to 6, only with the given $f(2) = 2$.