HOMEWORK QUIZ 1 SOLUTION

Due: Friday, Feb.3

1. *Supposing that A and B are two events in a random experiment. For each of the following two settings, determine whether A and B are independent, depend, or there is insufficient given information.*

(1) $\mathbb{P}(A) = 0.6$, $\mathbb{P}(B) = 0.8$;
(2) $\mathbb{P}(A) = 0.8$, $\mathbb{P}(B) = 0.2$, $\mathbb{P}(\textit{not } A \textit{ nor } B) = 0.16$.

Solution. (1) There is not sufficient information, since $\mathbb{P}(A \cap B)$ is not given nor implied. (2) If you draw a diagram, you could notice that $\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B)+\mathbb{P}(\text{not } A \text{ nor } B) = 1$. Applying the given numerical values gives $\mathbb{P}(A \cap B) = 0.16$, which is equal to $\mathbb{P}(A)\mathbb{P}(B)$. Thus, by definition A and B are independent random events.

2. *Assume that you are a game provider, the game being simply rolling a die once, then flipping a coin once. Both the die and the coin are fair. If the coin turns out "head", the gamer is rewarded x dollars, where x is the number showing on the die; otherwise, the gamer receives nothing. Write down the sample space. Let X be the random variable standing for the reward to the gamer after one game. What is the value of $\mathbb{E}[X]$? How much would you charge for each play in order to make an expected profit of \$1 per game?*

Solution. Sample space:

$$\{(1,H),(1,T),(2,H),(2,T),(3,H),(3,T),(4,H),(4,T),(5,H),(5,T),(6,H),(6,T)\}.$$

The random variable takes values 0 with probability $1/2$, and $1, 2, 3, 4, 5, 6$, each with nonzero probability $1/12$. Thus the expectation of X equals to

$$\mathbb{E}[X] = 0 \times \frac{1}{2} + (1 + 2 + ... + 6) \times \frac{1}{12} = \frac{7}{4} = 1.75.$$

Thus you'll need to charge 2.75 dollars for each game for the desired profit.

3. *For which values of x does the series $\sum_{n=1}^{\infty} e^{nx} = e^x + e^{2x} + e^{3x} + ...$ converge?*

Solution. Note that this is a geometric series with $r = e^x$. Such a geometric series converges if and only if $|r| < 1$. Now e^x could only take positive values, and $e^x < 1$ if and only if $x < 0$. We conclude that the given series converges if and only if $x < 0$.

4. *Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$, what is the limit of the series $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + ...$? Then, what is the limit of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$?*

Solution. In order, denote the terms of the three series given in question by a_n, b_n, c_n, respectively. We see that $b_n = \frac{1}{4}a_n$, and $c_n = a_n - b_n$. Therefore, by a fact on p.167 of

CoursePack, we have

$$\sum_{n=1}^{\infty} b_n = \frac{1}{4}\sum_{n=1}^{\infty} a_n = \frac{\pi^2}{24},$$

and

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = \frac{\pi^2}{8}.$$

5. *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers, that is, there exists a (finite) real number A, such that $\lim_{n\to\infty} a_n = A$. Show that the series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ converges by analyzing the limiting behavior its N-th partial sum. What is the limit of this series? (Note: See, for example, what the series looks like when $a_n = \frac{1}{n}$, or when $a_n = 7^{\frac{1}{n}}$, etc.)*

Solution. Let $S_N = \sum_{n=1}^{N}(a_{n+1} - a_n)$. It is straight-forward calculation (noting the massive cancellations happening between terms that are next to each other) that

$$S_N = a_{N+1} - a_1.$$

Moreover, since $a_n \to A$ as $n \to \infty$, we have that

$$\lim S_{N\to\infty} S_N = A - a_1.$$

Therefore, the original series converges to $A - a_1$.

6. *Use the comparison test to show that the series $\sum_{n=2}^{\infty} \frac{5+\sqrt{n}}{n^2+1}$ converges.*

Solution. Note that for all $n > 25$, we have that $\sqrt{n} > 5$; and $n^2 + 1 > n^2$. Thus for all those n's, we have

$$\frac{5+\sqrt{n}}{n^2+1} < \frac{2\sqrt{n}}{n^2} = \frac{2}{n^{3/2}}.$$

Because the series $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ converges (as a p-series), and *because each term of the original series is positive*, we have that the original series converges, by the comparison test.

7. *Use the integral test to show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.*

Solution. Note that the terms of the given series $\frac{1}{n \ln n}$ equals to $f(n)$, where

$$f(x) = \frac{1}{x \ln x}$$

is positive-valued and decreasing for $x \geq 2$. Moreover,

$$\int_2^{\infty} f(x)dx = \lim_{b\to\infty}\left(\ln(\ln x)\Big|_2^b\right) = \infty.$$

By the integral test, the convergence of the given series must be the same as that of the integral $\int_2^{\infty} f(x)dx$, hence divergent.

8. *What is the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$?*

Solution. By the ratio test, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{e}\frac{(n+1)^3}{n^3} = \frac{1}{e} < 1.$$

Therefore, the series converges.

9. *What are the lower and upper bounds incurred by taking the first 10 terms of the convergent series* $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$*?*

Solution. First note that the error incurred by taking S_{10} is precisely

$$E_{10} = \sum_{n=11}^{\infty} \frac{n}{(n^2+1)^2}.$$

Note that, corresponding to the series (with the integral test in mind), $f(x) = \frac{x}{(x^2+1)^2}$. Thus,

$$\int_{11}^{\infty} f(x)dx \leq E_{10} \leq \int_{10}^{\infty} f(x)dx,$$

which, by using $\int f(x)dx = -\frac{1}{2}\frac{1}{x^2+1}$, gives

$$\frac{1}{244} = -\frac{1}{2}\frac{1}{x^2+1}\Big|_{11}^{\infty} \leq E_{10} \leq -\frac{1}{2}\frac{1}{x^2+1}\Big|_{10}^{\infty} = \frac{1}{202}.$$

Bonus. *Does the series* $\sum_{n=1}^{\infty} \sin n$ *converge? Give your reasoning.*

Solution. No. For any positive integer k, and for any $n \in (2k\pi + \frac{1}{6}\pi, 2k\pi + \frac{5\pi}{6})$, we have $\sin n > \frac{1}{2}$. I show that there are infinitely many n such that there exists an k satisfying

$$n \in (2k\pi + \frac{1}{6}\pi, 2k\pi + \frac{5\pi}{6}).$$

For any k, there must exist a largest n that is less than $2k\pi + \frac{1}{6}\pi$, then $n+1$ must lie in the interval $(2k\pi + \frac{1}{6}\pi, 2k\pi + \frac{5\pi}{6})$, since $\frac{4\pi}{6} > 1$. This shows that there are infinitely many n such that $\sin n > 1/2$. By the n-th term test, the given series diverges.

Relevant text pages or hints:

1. CoursePack pp.140-141: independence of random events.
2. CoursePack p.145: random variables; coursePack p.153: expectation.
3. CoursePack p.163: geometric series.
4. CoursePack p.167: theorem 1.
5. CoursePack p 162: N-th partial sum; pp.163-164: "telescoping" series.
6. Note that for $n > 25$, one has $5 < \sqrt{n}$; also note that $1 + n^2 > n^2$.
7. Note that $\frac{d}{dx}\ln(\ln x) = \frac{1}{x \ln x}$ $(x > 0)$.
8. Ratio test.
9. CoursePack pp. 178-179. Error bound with integral test.
Bonus. n-th term test, but you'll need to be able to convince yourself that, for example, $\sin n$ attains values $> \frac{1}{2}$ infinitely many times; this latter argument being a little outside the scope of the current course, but not impossible if you realize that $7 - 2\pi \approx 0.72 < 2\pi/3$.