Geometry of Bäcklund Transformations

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I. Motivation

1. Cauchy-Riemann (CR):

$$\begin{cases} u_x - v_y = 0 \\ u_y + v_x = 0 \end{cases}$$

$\partial_x(Eq.1) + \partial_y(Eq.2) \Rightarrow \Delta u = 0$

$\partial_x(Eq.2) - \partial_y(Eq.1) \Rightarrow \Delta v = 0$

Given any harmonic function $u(x, y)$ (resp. $v(x, y)$), substituting it in (CR), the system for $v(x, y)$ (resp. $u(x, y)$) is completely integrable (aka. Frobenius). In particular, solutions can be found by solving ODEs only).

e.g. $u = x^2 - y^2 \Rightarrow v = 2xy + C$

(CR) is a way to 'relate' harmonic functions in 2 variables.

2. sine-Gordon Transformation (SGT):

$$\begin{cases} u_x - v_x = \lambda \sin(u + v) \\ u_y + v_y = \lambda^{-1} \sin(u - v) \end{cases}$$

($\lambda \neq 0$ const.)

$\partial_y(Eq.1) + \partial_x(Eq.2) \Rightarrow u_{xy} = \frac{1}{2}\sin(2u)$

$\partial_x(Eq.2) - \partial_y(Eq.1) \Rightarrow v_{xy} = \frac{1}{2}\sin(2v)$

Given any $u(x, y)$ (resp. $v(x, y)$) satisfying the sine-Gordon (SG) Equation

$$z_{xy} = \frac{1}{2}\sin(2z),$$

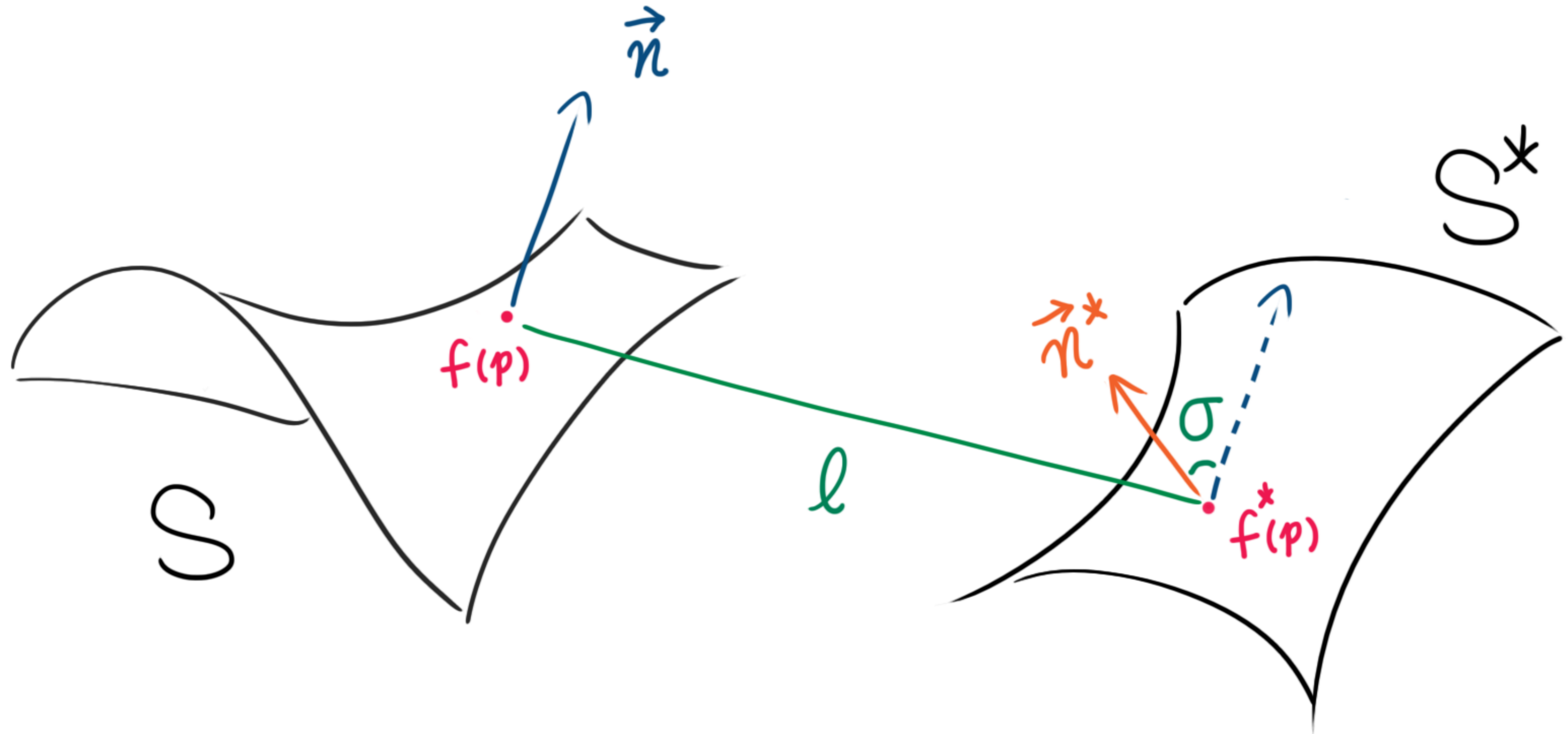
substituting it in (SGT), the system for $v(x, y)$ (resp. $u(x, y)$) is completely integrable.

e.g. $u = 0 \;\Rightarrow v = \arctan\left(C \exp\left(-\lambda x - \lambda^{-1} y\right)\right)$

(1-solitons)

(SGT) is a way to 'relate' solutions of (SG).

3. Pseudo-spherical Line Congruence (PSLC) in $\mathbb{E}^3$:



Two immersions $f, f^* : U \hookrightarrow \mathbb{E}^3$ of a surface are said to admit a *pseudo-spherical line congruence* between them if, for any $p \in U \subset \mathbb{R}^2$ (assuming $f(p) \neq f^*(p)$),

- The straight line $f(p)f^*(p)$ is tangent to both surfaces at $f(p)$ and $f^*(p)$, resp.;
- $d_{\mathbb{E}^3}(f(p), f^*(p)) = \ell > 0$ is a constant;
- The angle σ between the normals $\mathbf{n}(p)$ and $\mathbf{n}^*(p)$ is a constant in $(0, \pi)$.

Classical Bäcklund Theorem (Bianchi, Bäcklund, 1890s)

a) Two immersed surfaces $S, S^* \subset \mathbb{E}^3$ admit a pseudo-spherical line congruence (with parameters ℓ and σ) between them only if both S and S^* have the negative constant Gauss curvature:

$$K = -\frac{\sin^2 \sigma}{\ell^2}$$

b) For any immersed surface $S \subset \mathbb{E}^3$ with a negative constant Gauss curvature $K = -\ell^{-2} \sin^2 \sigma$ (for some fixed $\ell > 0$, $\sigma \in \mathbb{R}$), one can construct, by solving ODEs only, a 1-parameter family of $S^* \subset \mathbb{E}^3$ such that S and S^* are related by a pseudo-spherical line congruence with parameters ℓ and σ

Classical Bäcklund Theorem — A Closer Look

$\mathcal{F}$: (oriented) orthonormal frame bundle of $\mathbb{E}^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \in \mathcal{F} \subset \mathrm{SL}(4, \mathbb{R})$$

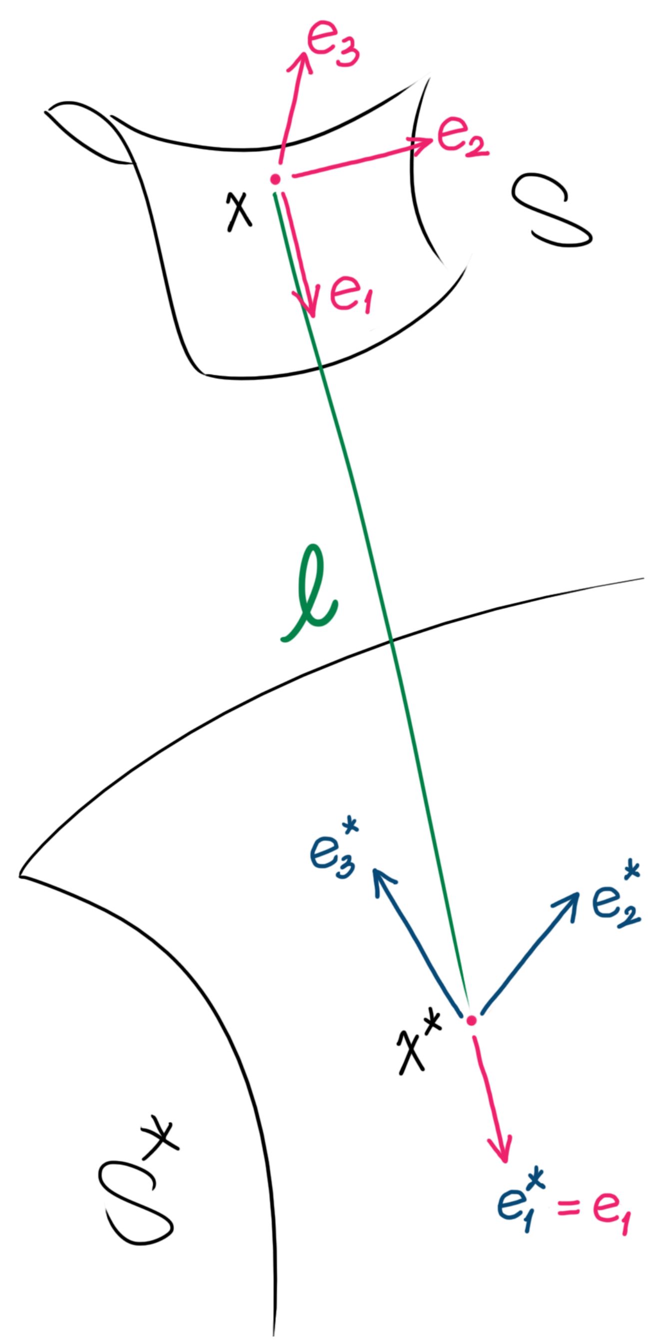
$$\mathbf{x}, \mathbf{e}_i \in \mathbb{R}^3; \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathrm{d}\mathbf{x} = \mathbf{e}_i \omega^i$$
$$\mathrm{d}\mathbf{e}_i = \mathbf{e}_j \omega_i^j$$

surface with $\mathbf{e}_3$ as normal $\Leftrightarrow$ $\omega^3 = 0$

Pseudo-spherical Line Congruence:

- can choose $\mathbf{e}_1$ s.t. $\overrightarrow{\mathbf{x}\mathbf{x}^*} = \ell \mathbf{e}_1$
- $\begin{cases} \mathbf{x}^* & = \mathbf{x} + \ell \mathbf{e}_1 \\ \mathbf{e}_1^* & = \mathbf{e}_1 \\ \mathbf{e}_2^* & = \mathbf{e}_2 \cos\sigma + \mathbf{e}_3 \sin\sigma \\ \mathbf{e}_3^* & = -\mathbf{e}_2 \sin\sigma + \mathbf{e}_3 \cos\sigma \end{cases}$
- (PSLC) $\omega^3 = \omega^{*3} = 0$



(PSLC): $\omega^3 = \omega^{*3} = 0$

(solutions are immersed surfaces in $\mathcal{F}$)

$$\left.\begin{aligned} \omega^3 &= 0 \\ \omega^{*3} &= 0 \\ d\omega^3 &= 0 \\ d\omega^{*3} &= 0 \end{aligned}\right\} \Rightarrow \left\{ \begin{aligned} \omega^1_3 \wedge \omega^2_3 &= -\frac{\sin^2\sigma}{\ell^2}\omega^1 \wedge \omega^2 \\ {\omega^*}^1_3 \wedge {\omega^*}^2_3 &= -\frac{\sin^2\sigma}{\ell^2}\omega^{*1} \wedge \omega^{*2} \end{aligned}\right.$$

$$\text{i.e. } K = K^* = -\frac{\sin^2\sigma}{\ell^2}$$

On the other hand, if $\iota : U \hookrightarrow M \cong \mathbb{R}^3 \times \mathbb{S}^2$ satisfies

$$\omega^3 = 0$$

$$\Upsilon_2 + \frac{\sin^2\sigma}{\ell^2}\Upsilon_0 = 0$$

(This is the equation for $K = -\sin^2\sigma/\ell^2$.)

then, restricted to $\pi^{-1}(\iota(U))$, the system (PSLC) is completely integrable.

Note:

$$\begin{array}{ccccccc} \pi^{-1}(\iota(U)) & \subset & \mathcal{F}^6 & \ni & (\mathbf{x}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \\ \downarrow & & \downarrow \pi & & \downarrow \\ U & \xhookrightarrow{\iota} & M^5 & \ni & (\mathbf{x}, \mathbf{e}_3) \end{array}$$

For an immersed surface $U \hookrightarrow M$ with $\mathbf{e}_3$ being the unit normal, on U,

$$\Upsilon_0 := \frac{1}{2}\mathbf{e}_3 \cdot (d\mathbf{x} \times d\mathbf{x}) = \omega^1 \wedge \omega^2$$

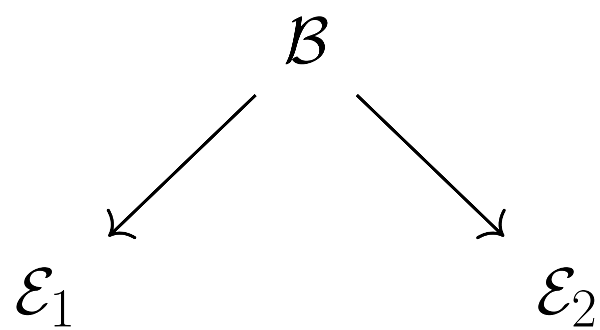
$$\Upsilon_1 := \frac{1}{2}\mathbf{e}_3 \cdot (d\mathbf{e}_3 \times d\mathbf{x}) = -H\omega^1 \wedge \omega^2$$

$$\Upsilon_2 := \frac{1}{2}\mathbf{e}_3 \cdot (d\mathbf{e}_3 \times d\mathbf{e}_3) = \omega^1_3 \wedge \omega^2_3 = K\omega^1 \wedge \omega^2$$

where H, K are the mean curvature and the Gauss curvature, resp.

In the 3 examples above...

$\mathcal{B}$	$\mathcal{E}_1$	$\mathcal{E}_2$
(CR)	$\Delta u = 0$	$\Delta v = 0$
(SGT)	(SG)	(SG)
(PSLC)	$(K = -\ell^{-2}\sin^2\sigma)$	$(K = -\ell^{-2}\sin^2\sigma)$



• $\mathcal{E}_i$ $(i = 1, 2)$ arise as integrability conditions of $\mathcal{B}$;

• Given any solution of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$), substituting it in $\mathcal{B}$, we obtain a completely integrable system whose solutions satisfy $\mathcal{E}_2$ (resp. $\mathcal{E}_1$).

Classically, $(\mathcal{B}; \mathcal{E}_1, \mathcal{E}_2)$ is called a Bäcklund transformation relating solutions of PDE systems $\mathcal{E}_1$ and $\mathcal{E}_2$.

(If) $\mathcal{E}_1$ and $\mathcal{E}_2$ are contact equivalent $\rightsquigarrow$ 'auto-Bäcklund transformation'

II. A Brief History

1883 **classical Bäcklund Theorem**; Bäcklund, Bianchi, Darboux, É.Cartan ...

1909 Tzitzeica equation (affine geometry) and the Tzitzeica transformation

1925 E. Goursat proposing/studying the **Bäcklund problem:**

Find all pairs of PDE systems whose solutions are related by a Bäcklund transformation.

1960 – more examples (e.g., auto-Bäcklund for KdV/Wahlquist, Estabrook, 1973)

affine analogue of the classical theorem/Chern, Terng, 1980

higher-dimensional generalization of the classical theorem/Tenenblat, Terng, 1980

soliton theory, integrable systems, loop groups (e.g., paper by Terng, Uhlenbeck, 2000)

2000 – geometric view using *Exterior Differential Systems* (Clelland, Ivey, I.M.Anderson, Fels ...)

Books include:

R. L. Anderson, N. H. Ibragimov, SIAM, 1979

C. Rogers, W. F. Schadwick, Academic Press, 1982

C. Rogers, W. K. Schief, Cambridge University Press, 2002

III. Geometric Formulation

Exterior Differential System (EDS)

(ref. book by Bryant et al.)

Def. An EDS is a pair $(M, \mathcal{I})$:

- M: smooth manifold
- $\mathcal{I}$: ideal in $\Omega^*(M)$ closed under d

Example 1

PDE $\quad z_{xy} = \sin z$

EDS $\quad (M, \mathcal{I})$

$M = J^1(\mathbb{R}^2, \mathbb{R})$

coordinates: (x, y, z, p, q)

$$\begin{aligned}\mathcal{I} = \langle & \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ & \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q, \\ & (\mathrm{d}p - \sin z\, \mathrm{d}y) \wedge \mathrm{d}x \rangle_{\mathrm{alg}}\end{aligned}$$

Def. Given an EDS $(M, \mathcal{I})$, an integral manifold is an immersed

$$\phi : U \to M$$

such that $\phi^*(\mathcal{I}) = 0$.

Example 2 ($K = -1$ system)

$M = \mathbb{R}^3 \times \mathbb{S}^2 \ni (\mathbf{x}, \mathbf{e}_3) \qquad \omega^3 := \mathrm{d}\mathbf{x} \cdot \mathbf{e}_3$

$\mathcal{I} = \langle \omega^3, \mathrm{d}\omega^3, \Upsilon_2 + \Upsilon_0 \rangle_{\mathrm{alg}}$

Theorem (Frobenius)

$(M, \mathcal{I})$ where $\mathcal{I}$ is locally algebraically generated by k linearly independent 1-forms $\theta_1, \ldots, \theta_k$

$$\Downarrow$$

There exist (by solving ODEs) local coordinates $(x^1, \ldots, x^{n-k}, y^1, \ldots, y^k)$ such that $\mathcal{I} = \langle \mathrm{d}y^1, \ldots, \mathrm{d}y^k \rangle_{\mathrm{alg}}$

(or: *'the distribution defined by $\theta_i = 0$ is completely integrable', 'locally M is foliated by a k-parameter family of integral manifolds', etc.*)

Integrable Extension

Def. An integrable extension of an EDS $(M^n, \mathcal{I})$ is an EDS $(N, \mathcal{J})$ with a submersion:

$$\pi : (N^{n+k}, \mathcal{J}) \to (M^n, \mathcal{I})$$

satisfying: locally $\exists$ 1-forms $\theta^1, ..., \theta^k$ on N, s.t.

$\mathcal{J} = \langle \pi^* \mathcal{I}, \theta^1, ..., \theta^k \rangle_{\text{alg}}$

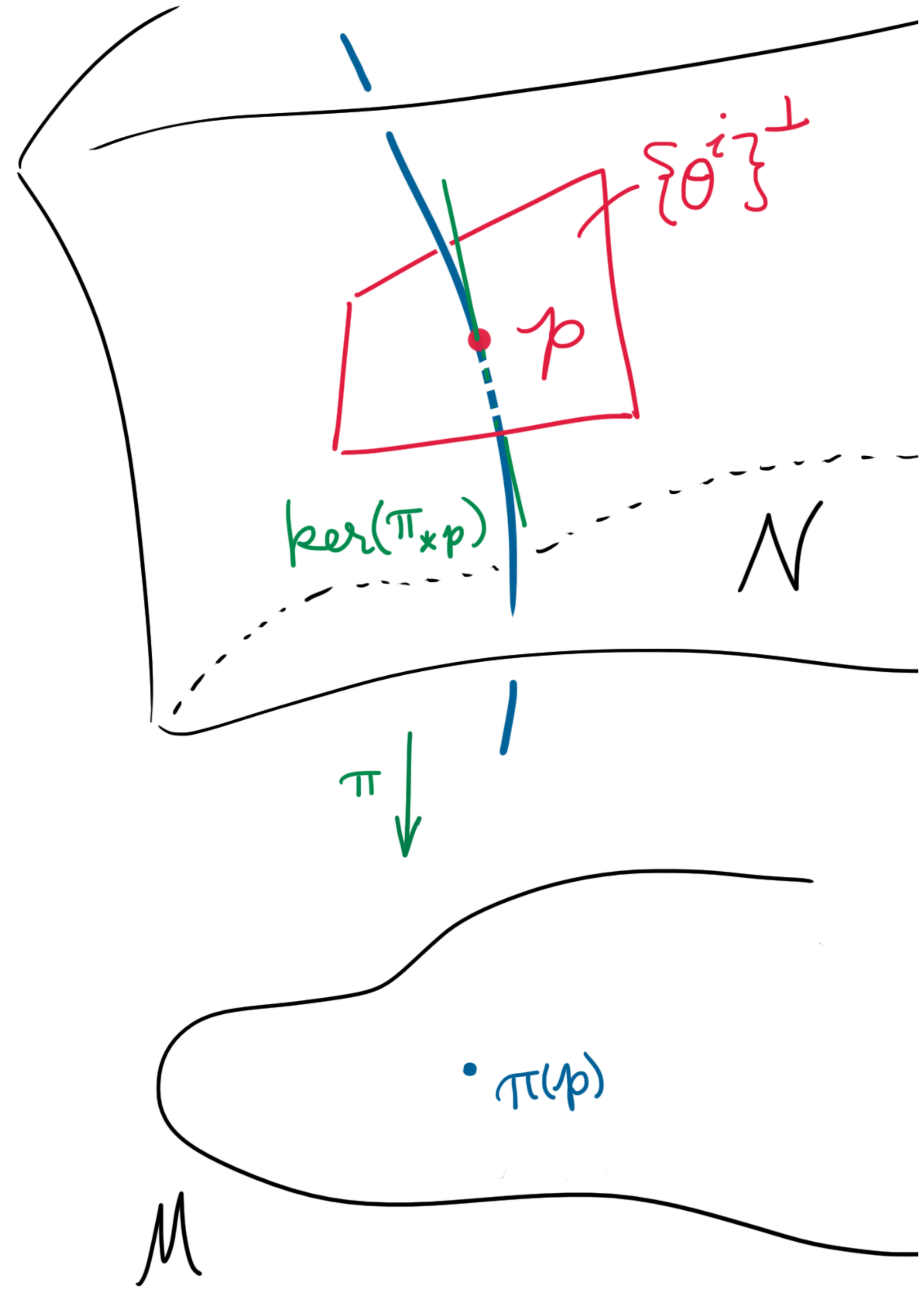
$\{\theta^1, ..., \theta^k\}^\perp \cap \ker(\pi_*) = 0$ (transversality)

Remark

a) if $S \subset M$ is an integral manifold, then $\mathcal{J}|_{\pi^{-1}S}$ is (rank-k) Frobenius

b) restricting to each integral manifold of $\mathcal{J}$, π is an immersion, the image being an integral manifold of $(M, \mathcal{I})$

(or: *'$\mathcal{J}$ pulls back to each such $\pi^{-1}S$ to define a flat connection'*)



Bäcklund Transformation

Def. A Bäcklund transformation (BT) relating two EDS $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$ is a quadruple $(N, \mathcal{B}; \pi_1, \pi_2)$:

$$\begin{array}{ccc} & (N, \mathcal{B}) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (M_1, \mathcal{I}_1) & & (M_2, \mathcal{I}_2) \end{array}$$

where both π_1 and π_2 are integrable extensions.

More terminology

A BT is ...

- *rank-k* if $\dim(M_1) = \dim(M_2) = n$ and $\dim(N) = n + k$
- *homogeneous* if the symmetry group of $(N, \mathcal{B})$ acts (locally) transitively on N

IV. Classification and Generality

Assumptions

- $(N, \mathcal{B}; \pi_1, \pi_2)$ has rank-1
- Both $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$ are hyperbolic Monge-Ampère (in the plane)

Hyperbolic Monge-Ampère (MA) Systems

	PDE	EDS
MA	$A(u_{xx}u_{yy} - u_{xy}^2) + Bu_{xx} + 2Cu_{xy} + Du_{yy} + E = 0$ $A, B, ..., E$ functions in x, y, u, u_x, u_y	$(M^5, \mathcal{I})$ $\mathcal{I} = \langle \theta, \mathrm{d}\theta, \Phi \rangle_{\text{alg}}$ θ contact form, $\Phi \in \Omega^2(M)$
Hyperbolic	$AE - BD + C^2 > 0$	$(\lambda \mathrm{d}\theta + \mu\Phi)^2 \equiv 0 \mod \theta$ has two distinct sol. $[\lambda : \mu] \in \mathbb{RP}^1$

Examples

$$u_{xy} = F(x, y, u, u_x, u_y) \qquad M = \mathbb{R}^3 \times \mathbb{S}^2,\ K = -1 \text{ in } \mathbb{E}^3$$

$$u_{xx}u_{yy} - u_{xy}^2 = -1 \qquad M = \mathbb{R}^3 \times \mathcal{H}^2,\ \text{timelike } K = 1 \text{ in } \mathbb{E}^{2,1}$$

$$\cdots \qquad \cdots$$

Theorem (Clelland, 2001)

Up to diffeomorphism, a homogeneous rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems belongs to one of the following **15** cases (in Clelland's numbering):

(1) auto-BT of $z_{xy} = 0$

(2) degenerate case

(3) between $z_{xy} = \exp(z)$ and $z_{xy} = 0$

(15) auto-BT between certain surfaces in some $M^5 = \mathrm{SO}^*(4)/\sim$

The rest are auto-BT relating surfaces in certain 3D Riemannian/Lorentzian space forms, where all prescribed curvatures are constants:

(4)	$K < 0$	$\mathbb{E}^3$	(7)	spacelike $K > 0$	$\mathbb{E}^{2,1}$	(11)	timelike $K > 1$	$\mathbb{S}^{2,1}$
(5)	$0 < K < 1$	$\mathbb{S}^3$	(8)	timelike $K > 0$	$\mathbb{E}^{2,1}$	(12)	spacelike $K > -1, \neq 0$	$\mathbb{H}^{2,1}$
(6)	$K < -1$	$\mathbb{H}^3$	(9)	timelike $H = 0$	$\mathbb{E}^{2,1}$	(13)	timelike $\lvert H\rvert > 1$	$\mathbb{H}^{2,1}$
			(10)	spacelike $K > 1$	$\mathbb{S}^{2,1}$	(14)	timelike $\lvert H\rvert \leq 1$	$\mathbb{H}^{2,1}$

Question: Without assuming homogeneity, how general are the rank-1 Bäcklund transformations being considered?

Main Theorem (H–, 2018)

A generic rank-1 BT relating two hyperbolic MA systems can be *uniquely* determined (up to diffeomorphism) by specifying at most **6** functions of **3** variables.

Corollary

There exist hyperbolic MA systems that are not related to any hyperbolic MA system by a generic rank-1 BT.

Proof of Corollary. To uniquely specify a hyperbolic MA system up to diffeomorphism, one needs **3** functions of **5** variables, which is much more than the data needed to specify a generic rank-1 BT being considered. □

Open Question. Given an arbitrary hyperbolic MA system, decide whether it admits a 'rank-1 Bäcklund pair'.

Outline of Proof

$$\begin{array}{ccc} & (N, \mathcal{B}) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (M_1, \mathcal{I}_1) & & (M_2, \mathcal{I}_2) \end{array}$$

$(M_i, \mathcal{I}_i)$ hyperbolic MA

$(N, \mathcal{B}; \pi_1, \pi_2)$ rank-$1 \Rightarrow \dim N = 6$

Step 1. all information is contained in $(N, \mathcal{B})$

$(N, \mathcal{B})$ is a rank-1 BT $\Leftrightarrow N$ admits local coframings of a certain type

Step 2. admissible coframes form a G-structure $\mathcal{G}$ associated to $(N, \mathcal{B})$

Step 3. assuming genericity, obtain local invariants of of the G-structure

Step 4. apply É.Cartan's generalization of Lie's 3rd theorem $\Rightarrow$ the generality bound

G-structure (Step 2)

$$\begin{array}{ccc} \mathrm{GL}(6, \mathbb{R}) & \longrightarrow & \mathcal{F} \\ & & \downarrow \pi \\ & & N^6 \end{array} \qquad \begin{array}{ccccc} G & \longrightarrow & \mathcal{G}^{14} & \subset & \mathcal{F} \\ & & \downarrow \pi & & \\ & & N^6 & & \end{array}$$

$\mathcal{F}$: the coframe bundle of N

$\mathcal{F}_p \ni u = (\eta^1, ..., \eta^6)^T : T_pN \xrightarrow{\cong} \mathbb{R}^6$

$G \subset \mathrm{GL}(6, \mathbb{R})$ subgroup consisting of

$$g = \begin{pmatrix} \det(\mathbf{B}) & 0 & 0 & 0 \\ 0 & \det(\mathbf{A}) & 0 & 0 \\ 0 & 0 & \mathbf{A} & 0 \\ 0 & 0 & 0 & \mathbf{B} \end{pmatrix}$$

$$\mathbf{A} = (a_{ij}),\ \mathbf{B} = (b_{ij})\ \in \mathrm{GL}(2, \mathbb{R})$$

Tautological 1-form on $\mathcal{G}$:

$$\omega := (\omega^1, \omega^2, ..., \omega^6)^T$$

G-structure Equations (Step 2, cont.) (Clelland, 2001)

$$\mathrm{d}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\\ \omega^6\end{pmatrix} = -\begin{pmatrix}\beta_0 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha_0 & 0 & 0 & 0 & 0\\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0\\ 0 & 0 & \alpha_3 & \alpha_0-\alpha_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \beta_1 & \beta_2\\ 0 & 0 & 0 & 0 & \beta_3 & \beta_0-\beta_1\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\\ \omega^6\end{pmatrix}$$

$$+\begin{pmatrix}A_1(\omega^3-C_1\omega^1)\wedge(\omega^4-C_2\omega^1)+\omega^5\wedge\omega^6\\ \omega^3\wedge\omega^4+A_4(\omega^5-C_3\omega^2)\wedge(\omega^6-C_4\omega^2)\\ B_1\omega^1\wedge\omega^2+C_1\omega^5\wedge\omega^6\\ B_2\omega^1\wedge\omega^2+C_2\omega^5\wedge\omega^6\\ B_3\omega^1\wedge\omega^2+C_3\omega^3\wedge\omega^4\\ B_4\omega^1\wedge\omega^2+C_4\omega^3\wedge\omega^4\end{pmatrix} \qquad (A_1, A_4\neq 0;\ A_1A_4\neq 1)$$

$$A_1(u\cdot g)=\frac{\det(\mathbf{A})}{\det(\mathbf{B})}A_1(u) \qquad A_4(u\cdot g)=\frac{\det(\mathbf{B})}{\det(\mathbf{A})}A_4(u)$$

$$\begin{pmatrix}B_1\\ B_2\end{pmatrix}(u\cdot g)=\det(\mathbf{AB})\mathbf{A}^{-1}\begin{pmatrix}B_1\\ B_2\end{pmatrix}(u) \qquad \begin{pmatrix}B_3\\ B_4\end{pmatrix}(u\cdot g)=\det(\mathbf{AB})\mathbf{B}^{-1}\begin{pmatrix}B_3\\ B_4\end{pmatrix}(u)$$

$$\begin{pmatrix}C_1\\ C_2\end{pmatrix}(u\cdot g)=\det(\mathbf{B})\mathbf{A}^{-1}\begin{pmatrix}C_1\\ C_2\end{pmatrix}(u) \qquad \begin{pmatrix}C_3\\ C_4\end{pmatrix}(u\cdot g)=\det(\mathbf{A})\mathbf{B}^{-1}\begin{pmatrix}C_3\\ C_4\end{pmatrix}(u)$$

Generic Case (Step 3)

Lemma. (H–)

Seeing G as acting on $\mathbb{R}^{10}$ (A_1, A_4, B_i, C_i), the maximal dimension of a G-orbit is **8**. Such maximal orbits, respectively, contain

$$\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} B_3 & C_3 \\ B_4 & C_4 \end{pmatrix} = \begin{pmatrix} \epsilon_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\epsilon_i = \pm 1$ $(i = 1, 2)$.

Def. $(N, \mathcal{B})$ is called generic if one of such maximal orbits is attained.

In the generic case, normalize B_i, C_i to one of the 4 cases above.

$\Rightarrow$ e-structure

- e-structrue has **42** local invariants, including A_1, A_4
- compatibility condition $\mathrm{d}^2 = 0 \Rightarrow$ **24** local invariants (as far as my calculation goes)

Cartan's Theorem (Step 4)

(ref. Bryant 2014 *Notes on EDS*, arXiv:1405.3116)

Cartan's generalization of Lie's 3rd theorem allows one to answer the question:

Do there exist linearly independent 1-forms $\omega^1, \ldots, \omega^n$ on M^n and functions

$$a = (a^\alpha) : M \to \mathbb{R}^s, \qquad b = (b^\rho) : M \to \mathbb{R}^r$$

such that

$$\mathrm{d}\omega^i = -\frac{1}{2} C^i_{jk}(a)\omega^j \wedge \omega^k, \qquad \mathrm{d}a^\alpha = F^\alpha_i(a,b)\omega^i,$$

where $C^i_{jk} = -C^i_{kj} : \mathbb{R}^s \to \mathbb{R}$ and $F^\alpha_i : \mathbb{R}^{s+r} \to \mathbb{R}$ are given and analytic?

In our situation, a^α are the invariants associated to the e-structure.

- Last Cartan character: $s_3 = 6$
- Involutive?
  No $\Rightarrow$ 'at most' **6** functions of **3** variables □

Note: Involutivity can be attained, according to Cartan-Kuranishi, if we further 'prolong' the structure equations. However, one would encounter an enormous amount calculation.

V. New Classification Results and Examples

Assumptions

A1) the generic case

A2) A_1, A_4, B_i, C_i take value in the G-orbit with $\epsilon_1 = \epsilon_2 = 1$

A3) $A_1 = 1$ and $A_4 = -1$

Results

Theorem (H–, 2018)

A rank-1 BT satisfying the assumptions A1)-A3) can be uniquely determined by specifying **1** function of **2** variables, up to diffeomorphism.

Rmk. The proof is similar to that of the Main Theorem. Here, the structure equations are involutive with the last Cartan character $s_2 = 1$.

In fact, A1)-A3) $\Rightarrow$ **2 subcases**

Case 1: homogeneous case

in Clelland's classification: (13) $H > 1$ in $\mathbb{H}^{2,1}$, (14) $0 < H \leq 1$ in $\mathbb{H}^{2,1}$ and (15)

Case 2: higher cohomogeneity

Proposition. (H–)

If $(N, \mathcal{B})$ is a rank-1 BT satisfying A1)-A3), then there exists a canonical map

$$\Phi = (R, S, T) : N \to \mathbb{R}^3$$

satisfying: Φ has

- rank 1 $\Leftrightarrow$ $2RS = 1$ and $T = R^2 + S^2$
- rank 2 $\Leftrightarrow$ not rank-1 and $\left(2RS = 1 \text{ or } \begin{cases} T_4 = (R+S)(T-1) \\ T_6 = (R-S)(T+1) \end{cases}\right)$

  (Note: $\mathrm{d}T = T_i \omega^i$)
- rank 3 $\Leftrightarrow$ neither rank-1 nor rank-2

Next, we'll focus on the rank-1 case and the highlighted rank-2 case.

Theorem. (H–, 2018) **When Φ has rank 1**, $(N, \mathcal{B})$

i) is of cohomogeneity-1

ii) is an auto-BT of the hyperbolic MA equation

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} = 0$$

where $A = 2z_x + y$ and $B = 2z_y - x$

iii) has Lie algebra of symmetry $\mathfrak{h}$

$\dim(\mathfrak{h}) = 5$

$\mathfrak{h}$ solvable and not nilpotent

derived series has dim. $(5, 3, 1, 0, \ldots)$

$\mathfrak{h}$ is isomorphic to the Lie algebra generated by the real and imaginary parts of

$$\partial_w, \qquad e^{2w}(\partial_z + i\bar{z}\partial_\lambda), \qquad e^{2(w+\bar{w})}\partial_\lambda$$

on $\mathbb{R} \times \mathbb{C}^2 \ni (\lambda, z, w)$

Theorem. (H–, 2018) **When Φ has rank 2 and**

$$\begin{cases} T_4 = (R+S)(T-1) \\ T_6 = (R-S)(T+1) \end{cases}$$

i) the image of Φ is contained in a surface defined by $[R^2 + S^2 - T : 2RS - 1] = \text{const.} \in \mathbb{RP}^1$ ($(N, \mathcal{B})$ has cohomogeneity-2)

ii) $(N, \mathcal{B})$ has Lie algebra of symmetry

$$\mathfrak{q} \cong \begin{cases} \mathfrak{so}(3) \oplus \mathbb{R} & \text{if } R^2 + S^2 < T \\ \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} & \text{if } R^2 + S^2 > T \\ \mathfrak{g} \oplus \mathbb{R} & \text{if } R^2 + S^2 = T \end{cases}$$

where $\mathfrak{g}$ is the solvable 3-dimensional Lie algebra with basis $\{x_1, x_2, x_3\}$ satisfying

$$[x_3, x_1] = x_2, \quad [x_2, x_3] = x_1, \quad [x_1, x_2] = 0$$

iii) when $R^2 + S^2 \equiv T$, $(N, \mathcal{B})$ is an auto-BT of the equation

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4AB z_{xy} + \epsilon(A^2 + B^2)^2 = 0$$

where $A = z_x - y$, $B = z_y + x$; and $\epsilon = \pm 1$