Absolute Equivalence: after Cartan and Sluis

(based on joint work with Jeanne N. Clelland)

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$1/9$. The Hilbert-Cartan ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)^2 \qquad (\star)$$

As a Pfaffian system (M, I)

$$M = \mathbb{R}^5 \ni (x, y, t, u, v) = (x, y, t, y', y'')$$

$$I = [\![\, \mathrm{d}y - u\mathrm{d}t, \quad \mathrm{d}u - v\mathrm{d}t, \quad \mathrm{d}x - v^2\mathrm{d}t \,]\!] \subset T^*M$$

Theorem (Hilbert, 1912) One **cannot** express the general solutions of $(\star)$ in the form

$$\begin{cases} t = T\left(\alpha, g, \dfrac{\mathrm{d}g}{\mathrm{d}\alpha}, \ldots, \dfrac{\mathrm{d}^k g}{\mathrm{d}\alpha^k}\right) \\ x = X\left(\alpha, g, \dfrac{\mathrm{d}g}{\mathrm{d}\alpha}, \ldots, \dfrac{\mathrm{d}^k g}{\mathrm{d}\alpha^k}\right) \\ y = Y\left(\alpha, g, \dfrac{\mathrm{d}g}{\mathrm{d}\alpha}, \ldots, \dfrac{\mathrm{d}^k g}{\mathrm{d}\alpha^k}\right) \end{cases}$$

T, X, Y determined functions; $g(\alpha)$ arbitrary function in α.

2/9. Sometimes you can ...

ODE

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 1$$

General solutions

$$\begin{cases} t = g''(\alpha) + g(\alpha) \\ x = g''(\alpha)\cos\alpha + g'(\alpha)\sin\alpha \\ y = g''(\alpha)\sin\alpha - g'(\alpha)\cos\alpha \end{cases}$$

... but when?

Given an ODE/system whose solutions depend on one function, how to tell whether its general solutions can be expressed in a determined way (in terms of one arbitrary function of one variable and its derivatives)? (Hilbert's question)

3/9. É. Cartan (1914) – The corank 2 case

(M, I): Pfaffian system with a degree-1 independence condition

Definition. A **Cartan prolongation** of (M, I) is a system (N, J) with a submersion

$$\pi : N \to M$$

such that

(1) $\pi^* I \subset J$

(2) any generic integral curve of (M, I) admits a **unique** lifting to be an integral curve of (N, J)

Cartan (1914) noticed: when $\dim(M) - \operatorname{rank}(I) = \mathbf{2} \ldots$

- **i.** any Cartan prolongation is a succession of **total prolongations**
- **ii.** a total prolongation is **'cancelled'** by computing the first derived system

  e.g.

$$[\![\mathrm{d}g - w\mathrm{d}t]\!] \xrightarrow{\mathrm{pr}^{(1)}(\cdots)} [\![\mathrm{d}g - w\mathrm{d}t,\ \mathrm{d}w - v\mathrm{d}t]\!] \xrightarrow{(\cdots)^{(1)}} [\![\mathrm{d}g - w\mathrm{d}t]\!]$$

4/9. Absolute equivalence

Definition. Two systems (M, I) and $(\bar{M}, \bar{I})$ are **absolutely equivalent** if there exists a system (N, J) and the following diagram in which π and $\bar{\pi}$ are both Cartan prolongations.

$$\begin{array}{ccc} & (N, J) & \\ {}^{\pi}\swarrow & & \searrow^{\bar{\pi}} \\ (M, I) & & (\bar{M}, \bar{I}) \end{array}$$

Hilbert's question rephrased: When is a system (M, I) absolutely equivalent to $[\![\mathrm{d}g - w\mathrm{d}t]\!]$ on $\mathbb{R}^3$?

Cartan's idea: In the corank 2 case, by successively computing the derived systems, one can turn a system into one that's not the result of any total prolongation, called the associated **normal system**.

Theorem (Cartan, 1914) Two **corank 2** systems are absolutely equivalent if and only if their normal systems are **contact** equivalent.

"The corank $k \geq 3$ case is difficult."

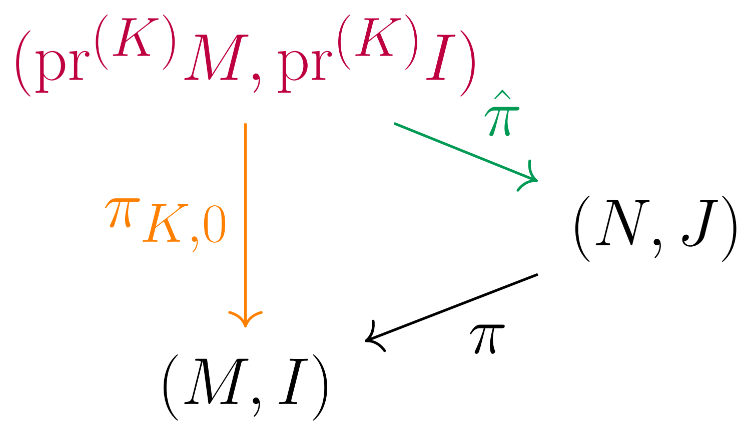
5/9. W. Sluis (1994) – Extension theorem

Prolongation by differentiation

e.g.

$$I = [\![\mathrm{d}x - u\mathrm{d}t,\ \mathrm{d}y - v\mathrm{d}t,\ \mathrm{d}z - (u^2 + v^2)\mathrm{d}t]\!] \rightsquigarrow [\![I,\ \mathrm{d}u - w\mathrm{d}t]\!] = J$$

Theorem (Sluis, 1994) Given any Cartan prolongation $\pi : (N, J) \to (M, I)$, one can construct a commutative diagram of the following form,



where $\pi_{K,0}$ stands for the K-th total prolongation, and $\hat{\pi}$ is a succession of prolongations by differentiation.

6/9. The corank 3 case

$\pi : (N, J) \to (M, I)$ Cartan prolongation. Determine the **relative extensions** I_k by

$$\begin{cases} I_0 = \pi^* I, \\ I_{k+1} = \mathcal{C}(I_k) \cap J, \quad k = 0, 1, \ldots \end{cases}$$

where $\mathcal{C}(\cdots)$ stands for the retraction system (vector bundle).

In general, $I_k \subset I_{k+1}$ needs **not** induce a Cartan prolongation.

Theorem (Clelland and H–, 2019)

$$\left\{ \begin{array}{l} (M, I) \text{ system of corank } \mathbf{3} \\ \\ \pi : (N, J) \to (M, I) \text{ a succession of} \\ \text{prolongations by differentiation} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} I_0 \subseteq \cdots \subseteq I_k \subseteq I_{k+1} \subseteq \cdots \text{ stabilizes at } J \\ \\ I_0 \subseteq \cdots \subseteq I_k \subseteq \cdots \subseteq J \text{ represents a number} \\ \text{of total prolongations, if any, followed by} \\ \text{rank-1 prolongations by differentiation} \end{array} \right\}$$

7/9. Systems of control type (CTS)

Definition. A system (M, I) with independence condition $\tau = \mathrm{d}t$ is called a **system of control type** (CTS) if $[\![I, \tau]\!]$ is Frobenius.

Note.

(1) Locally, one can choose coordinates to put a CTS in the form

$$I = [\![\mathrm{d}x^i - f^i(\boldsymbol{x}, \boldsymbol{u}, t)\mathrm{d}t]\!]_{i=1}^n$$

(2) A corank **2** system is always a CTS.

Lemma (Sluis, 1994) A CTS $(M, I; \tau)$ is **controllable and linear** (that is, it can be put in a Brunovský normal form by a coordinate-change) if and only if

i. $I^{(\infty)} = 0$

ii. each $[\![I^{(k)}, \tau]\!]$ is Frobenius

8/9. Dynamic feedback linearization

Question: *Which CTS are absolutely equivalent to a controllable, linear CTS?*

Theorem (Clelland and H–, 2019) A **corank 3** CTS $(M, I; \mathrm{d}t)$ is absolutely equivalent[1] to a controllable, linear CTS if and only if it admits a Cartan prolongation

$$\pi : (N, J; \mathrm{d}t) \to (M, I; \mathrm{d}t),$$

where

- **i.** $(N, J; \mathrm{d}t)$ is a controllable, linear CTS
- **ii.** each $I_k \subsetneq I_{k+1}$ represents a rank-1 prolongation by differentiation

Let $K := \mathrm{rank}(J) - \mathrm{rank}(I)$.

Question: *Fixing $(M, I; \mathrm{d}t)$, what is the* **minimum** *integer K that allows a Cartan prolongation as described in the theorem to exist?* ($K = 0$ means that $(M, I; \mathrm{d}t)$ is already controllable and linear.)

[1]. . . in a way that the independence conditions match each other.

9/9. Classification when $\dim(M) = 6$ and $\operatorname{rank}(I) = 3$

$\boxed{K = \mathbf{1}}$

$$\mathfrak{A}_{[1,1]} : \left\{ \begin{array}{l} \mathrm{d}z - p\mathrm{d}x, \\ \mathrm{d}x - v\mathrm{d}t, \\ \mathrm{d}p - w\mathrm{d}t \end{array} \right\}$$

$$\mathfrak{A}_{[1,2]} : \left\{ \begin{array}{l} \mathrm{d}z - p\mathrm{d}x, \\ \mathrm{d}p - v\mathrm{d}x, \\ \mathrm{d}x - w\mathrm{d}t \end{array} \right\}$$

$$\mathfrak{A}_{[2,2,3]} : \left\{ \begin{array}{l} -f(x,y,t,p,q)\mathrm{d}p - \mathrm{d}q + s\mathrm{d}t, \\ \mathrm{d}y - p\mathrm{d}x - q\mathrm{d}t, \\ \mathrm{d}x - f(x,y,t,p,q)\mathrm{d}t \end{array} \right\}$$

$$(f_p - f_q f > 0)$$

9/9. Classification when $\dim(M) = 6$ and $\operatorname{rank}(I) = 3$

$\boxed{K = \mathbf{2}}$

precisely when there exists a coframing $(\theta^1, \ldots, \theta^5, \tau)$ such that $I = [\![\theta^1, \theta^2, \theta^3]\!]$ and ...

$$\mathfrak{A}_{[3,1,1,3]} : \begin{cases} \mathrm{d}\theta^1 \equiv \tau \wedge \theta^4 + \theta^2 \wedge (Q_1\theta^3 + Q_2\theta^4) & \mod \theta^1, \\ \mathrm{d}\theta^2 \equiv \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4 & \mod \theta^2, \\ \mathrm{d}\theta^3 \equiv \tau \wedge \theta^5 & \mod \theta^3. \end{cases}$$

$$\mathfrak{A}_{[3,1,2,3]} : \begin{cases} \mathrm{d}\theta^1 \equiv \tau \wedge \theta^4 + \theta^3 \wedge \theta^5 + \theta^2 \wedge (Q_1\theta^3 + Q_2\theta^4 + Q_3\theta^5) & \mod \theta^1, \\ \mathrm{d}\theta^2 \equiv \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4 & \mod \theta^2, \\ \mathrm{d}\theta^3 \equiv \tau \wedge \theta^5 & \mod \theta^3. \end{cases}$$

9/9. Classification when $\dim(M) = 6$ and $\operatorname{rank}(I) = 3$

$\boxed{K = \mathbf{3}}$

$$\mathfrak{A}_{[2,1,2]} : \left\{ \begin{array}{l} \mathrm{d}z - y\mathrm{d}x, \\ \mathrm{d}y - w\mathrm{d}x, \\ \mathrm{d}w - p\mathrm{d}t - f(x,y,z,w,t)\mathrm{d}x \end{array} \right\}$$

$$\mathfrak{A}_{[3,1,1,2]} : \begin{cases} \mathrm{d}\theta^1 \equiv \tau \wedge \theta^4 + \theta^2 \wedge (Q_1\theta^3 + Q_2\theta^4) & \mod \theta^1, \\ \mathrm{d}\theta^2 \equiv \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4 & \mod \theta^2, \\ \mathrm{d}\theta^3 \equiv (\theta^2 + \tau) \wedge \theta^5 & \mod \theta^3. \end{cases}$$

$$\mathfrak{A}_{[3,1,2,2]} : \begin{cases} \mathrm{d}\theta^1 \equiv \tau \wedge \theta^4 + \theta^3 \wedge \theta^5 + \theta^2 \wedge (Q_1\theta^3 + Q_2\theta^4 + Q_3\theta^5) & \mod \theta^1, \\ \mathrm{d}\theta^2 \equiv \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4 & \mod \theta^2, \\ \mathrm{d}\theta^3 \equiv (\theta^2 + \tau) \wedge \theta^5 & \mod \theta^3. \end{cases}$$

9/9. Classification when $\dim(M) = 6$ and $\operatorname{rank}(I) = 3$

$\boxed{K = \mathbf{4}}$

Not a classification yet, but we have existence.

Example.

$$\left\{ \begin{array}{l} \mathrm{d}x - u\mathrm{d}t, \\ \mathrm{d}y - (uv + x(z - xv^2))\mathrm{d}t, \\ \mathrm{d}z - (uv^2 + x(y - xv))\mathrm{d}t \end{array} \right\}$$

$\boxed{K \geq \mathbf{5}}$ still mysterious

Necessary condition for a **finite** K: *When you write* $I = [\![\mathrm{d}x^i - f^i(\boldsymbol{x}, \boldsymbol{u}, t)\mathrm{d}t]\!]$*, for each fixed* $\boldsymbol{x}, t$

$$(u^1, u^2) \mapsto \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \in \mathbb{R}^3$$

must parametrize a **ruled** *surface.* (Pomet, 2009)

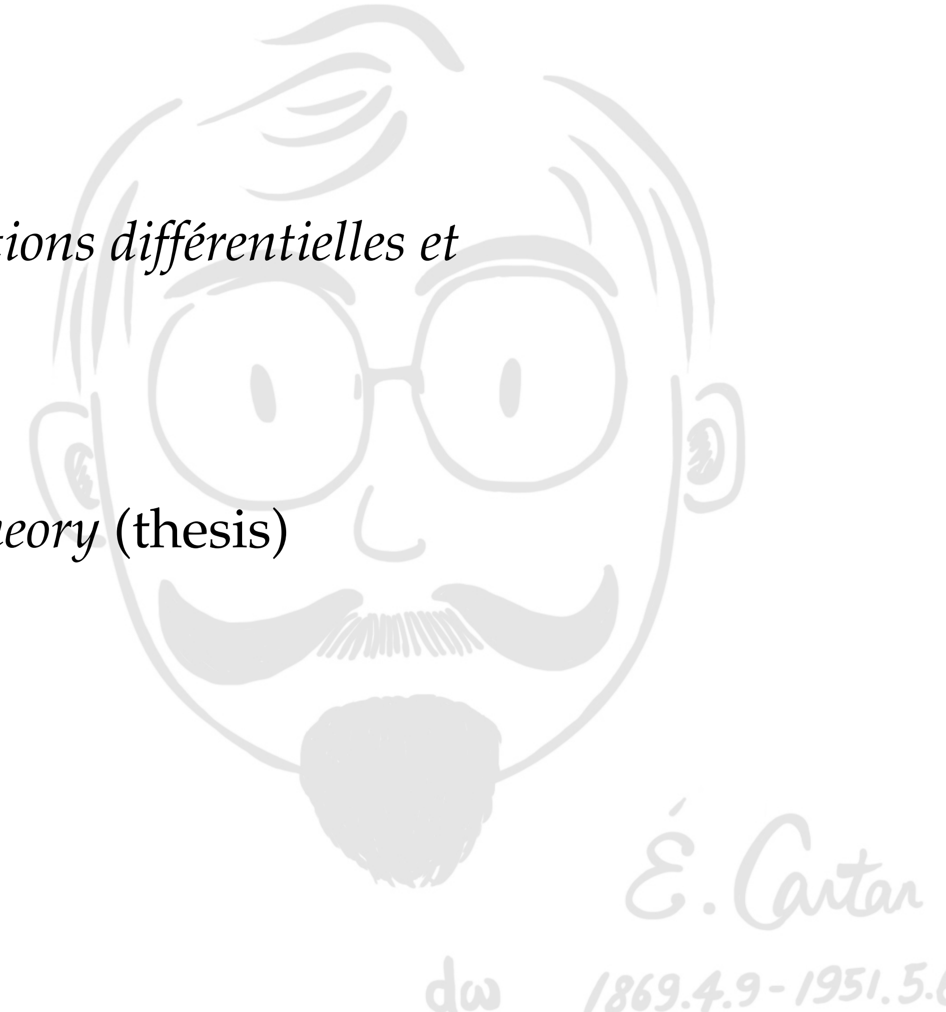
Either this condition is not sufficient or
there exists a CTS with finite minimum $K \geq 5$*, or both.* (Clelland and H–, 2019)

References

- É. Cartan (1914)
  Sur l'équivalence absolue de certains systemes d'équations différentielles et sur certaines familles de courbes

- W. Sluis (1994)
  Absolute equivalence and its applications to control theory (thesis)

- J.-B. Pomet (2009)
  A necessary condition for dynamic equivalence



Thank you!