GEOMETRY OF BÄCKLUND TRANSFORMATIONS II: MONGE-AMPÈRE INVARIANTS

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ABSTRACT. This paper is concerned with the question: *For which pairs of hyperbolic Euler-Lagrange systems in the plane does there exist a rank-1 Bäcklund transformation relating them?* We express some obstructions to such existence in terms of the local invariants of the Euler-Lagrange systems. In addition, we discover a class of Bäcklund transformations relating two hyperbolic Euler-Lagrange systems of distinct types.

CONTENTS

1. INTRODUCTION

A Bäcklund transformation is a way to relate solutions of two PDE systems $\mathcal{E}_1$ and $\mathcal{E}_2$, in such a manner that, given a solution of $\mathcal{E}_1$, one can use it to obtain solutions of $\mathcal{E}_2$ by solving ODEs, and *vice versa*. Early studies of Bäcklund transformations date back to the late 19th century.

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1.1. **Three Examples.** The following examples of Bäcklund transformations are classical.

1. Let $u = u(x, y)$ be a harmonic function. One can find a 1-parameter family of harmonic functions $v = v(x, y)$ by substituing $u(x, y)$ into the Cauchy-Riemann system in u and v and by solving ODEs. In this sense, the Cauchy-Riemann system is a Bäcklund transformation relating solutions of the Laplace equation $\Delta u(x, y) = 0$.

2. Let $u = u(x, y)$ be a solution of the *sine-Gordon* equation

$$u_{xy} = \frac{1}{2}\sin(2u). \tag{1}$$

Substituting $u(x, y)$ in the system (with parameter $\lambda \neq 0$) in u and v:

$$\begin{cases} u_x - v_x = \lambda \sin(u + v), \\ u_y + v_y = \lambda^{-1} \sin(u - v), \end{cases} \tag{2}$$

one obtains a compatible first-order PDE system in $v(x, y)$, which can be solved using ODE methods. In this case, the system (2) is a Bäcklund transformation relating solutions of the sine-Gordon equation. In particular, if we start by setting $u(x, y) = 0$, which is a solution of (1), solving (2) (with a fixed λ) for $v(x, y)$ yields a 1-parameter family of solutions of (1):

$$v(x, y) = \arctan\left(C \exp\left(-\lambda x - \lambda^{-1} y\right)\right),$$

where C is an arbitrary constant. Such $v(x, y)$ are known as 1-*soliton* solutions of (1). Iterative application of the Bäcklund transformation (2) yields the so-called n-*solitons* of (1). Interested reader can see [TU00] and [RS02] for more details.

3. Given two immersed surfaces $S_1, S_2 \subset \mathbb{E}^3$, by a *pseudo-spherical (p.s.) line congruence between them* we mean immersions $\sigma_1, \sigma_2 : U \to \mathbb{E}^3$ ($U \subset \mathbb{R}^2$ open) satisfying $\sigma_i(U) = S_i$ and

(1) The distance $d_{\mathbb{E}^3}(\sigma_1(p), \sigma_2(p))$ is a constant $r > 0$;
(2) For each $p \in U$, the line through $\sigma_1(p)$ and $\sigma_2(p)$ is tangent to both surfaces at $\sigma_1(p)$ and $\sigma_2(p)$, respectively;
(3) The respective normals $\boldsymbol{n}_1(p)$ and $\boldsymbol{n}_2(p)$ form a constant angle $\theta \in (0, \pi)$.

When these conditions hold, S_1, S_2 are called the *focal surfaces* of the corresponding p.s. line congruence.

It is a theorem of Bianchi that $S_1, S_2 \subset \mathbb{E}^3$ *admit a p.s. line conguence (with parameters r, θ) between them only if they both have the Gauss curvature $K = -\sin^2\theta/r^2$. Conversely, if $S \subset \mathbb{E}^3$ satisfies $K = -\sin^2\theta/r^2$ for some constants $r > 0$ and $\theta \in (0, \pi)$, then one can construct a 1-parameter family of p.s. line congruences with parameters r and θ and with S being a focal surface.*

In fact, when S_1, S_2 admit a p.s. line congruence with parameters r and θ, the Gauss and Codazzi equations of S_i ($i = 1, 2$) together form a PDE system $\mathcal{E}_{r,\theta}$. The integrability of $\mathcal{E}_{r,\theta}$ puts conditions on the respective Gauss curvatures. On the other hand, given a surface S with constant Gauss curvature $K = -\sin^2\theta/r^2 < 0$, the problem of finding a p.s. line congruence with parameters r, θ and with S as a focal surface reduces to integrating a Frobenius (aka. completely integrable) system. In this sense, $\mathcal{E}_{r,\theta}$ is a Bäcklund transformation relating surfaces with $K = -\sin^2\theta/r^2$. For more details, including how $\mathcal{E}_{r,\theta}$ relates to the sine-Gordon equation and its Bäcklund transformations, see [CT80] and [BGG03].

Using the theory of exterior differential systems, one can study Bäcklund transformations from a geometric viewpoint. For instance, the Bäcklund transformation $\mathcal{E}_{r,\theta}$ in Example **3** above is *homogeneous* in the sense that the symmetry group of the system acts locally transitively on the space of variables. It has *rank* 1 in the sense that it generates a 1-parameter family of surfaces with $K = -\sin^2\theta/r^2$ from a given one. The PDE system for S with a prescribed constant Gauss curvature $K < 0$ is an example of a *hyperbolic Monge-Ampère system.* A complete classification of rank-1 homogeneous Bäcklund transformations relating two hyperbolic Monge-Ampère systems has been obtained by Jeanne N. Clelland in [Cle02], using Cartan's method of equivalence.

1.2. **Geometric Formulations.** Classically, a *Monge-Ampère equation* in $z(x, y)$ is a second-order PDE of the form

$$A(z_{xx}z_{yy} - z_{xy}^2) + Bz_{xx} + 2Cz_{xy} + Dz_{yy} + E = 0, \tag{3}$$

where A, B, C, D, E are functions of x, y, z, z_x, z_y. This is a class of PDEs that are closely related to surface geometry and calculus of variations (see [Bry99] and [BGG03]). A Monge-Ampère equation (3) is said to be *hyperbolic* (resp., *elliptic*, *parabolic*) if $AE - BD + C^2$ is positive (resp., negative, zero).

Geometrically, each PDE system can be formulated as an *exterior differential system* (see [BCG$^+$13]). In particular, since the current work will mainly be concerned with hyperbolic Monge-Ampère systems, we make the definition below, following [BGH95].

Definition 1.1. A *hyperbolic Monge-Ampère system* is an exterior differential system $(M, \mathcal{I})$, where M is a 5-manifold; $\mathcal{I} \subset \Omega^*(M)$ is a differential ideal satisfying: for each $p \in M$, there exists an open neighborhood U of p on which $\mathcal{I}|_U$ is algebraically generated by

$$\theta \in \mathcal{I} \cap \Omega^1(U) \qquad \text{and} \qquad \mathrm{d}\theta, \Omega \in \mathcal{I} \cap \Omega^2(U);$$

these generators must satisfy:

(1) $\theta \wedge (\mathrm{d}\theta)^2 \neq 0$;
(2) pointwise, $\mathrm{span}\{\mathrm{d}\theta, \Omega\}$, modulo θ, has rank 2;
(3) $(\lambda \mathrm{d}\theta + \mu\Omega)^2 \equiv 0 \mod \theta$ has two distinct solutions $[\lambda_i : \mu_i] \in \mathbb{RP}^1$ $(i = 1, 2)$.

For example, the sine-Gordon equation (1) may be formulated as a hyperbolic Monge-Ampère system $(M, \mathcal{I})$ in the following way. Let $M \subset \mathbb{R}^5$ be an open domain with coordinates (x, y, u, p, q). Let $\mathcal{I} \subset \Omega^*(M)$ be the ideal generated by

$$\theta = \mathrm{d}u - p\mathrm{d}x - q\mathrm{d}y, \quad \mathrm{d}\theta = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q$$

and

$$\Omega = \left(\mathrm{d}p - \frac{1}{2}\sin(2u)\mathrm{d}y\right) \wedge \mathrm{d}x.$$

It is easy to see that a surface $\iota : S \hookrightarrow M$ satisfying $\iota^*(\mathrm{d}x \wedge \mathrm{d}y) \neq 0$ corresponds to a solution of (1) if and only if $\iota^*\mathcal{I} = 0$. Moreover, the three conditions in Definition 1.1 hold.

In general, let $(M, \mathcal{I})$ be a hyperbolic Monge-Ampère system. It is easy to show that, each point $p \in M$ has an open neighborhood U on which there exist 1-forms $\theta, \omega^1, \omega^2, \omega^3, \omega^4$, everywhere linearly independent, such that[1]

$$\mathcal{I}|_U = \langle \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle_{\mathrm{alg}}.$$

Such a list of 1-forms is called a 0-*adapted (local) coframing of* $(M, \mathcal{I})$.

Suppose that $(\theta, \omega^1, \cdots, \omega^4)$ is a 0-adapted local coframing of a hyperbolic Monge-Ampère system $(M, \mathcal{I})$. Let $\iota : S \hookrightarrow M$ be an embedded surface satisfying $\iota^*\mathcal{I} = 0$ and the independence condition $\iota^*(\omega^1 \wedge \omega^3) \neq 0$. Such a surface is called an *integral manifold* (or, in this case, an *integral surface*) of $(M, \mathcal{I})$. On S, the 1-forms ω^1 and ω^2 are multiples of each other; so are ω^3 and ω^4. The equations $\omega^1 = 0$ and $\omega^3 = 0$ each define a tangent line field on S. They are just the classical characteristics of the hyperbolic PDE corresponding to $(M, \mathcal{I})$.

Definition 1.2. Given a hyperbolic Monge-Ampère system[2] $(M, \mathcal{I})$, where $\mathcal{I} = \langle \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle$, the pair of Pfaffian systems $\mathcal{I}_{10} = \langle \theta, \omega^1, \omega^2 \rangle$ and $\mathcal{I}_{01} = \langle \theta, \omega^3, \omega^4 \rangle$, which are defined up to ordering, are called the *characteristic systems* associated to $(M, \mathcal{I})$.

We follow [BGG03] to give the following definition.[3]

Definition 1.3. Given two hyperbolic Monge-Ampère systems $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$ with $\mathcal{I}_i = \langle \theta_i, \mathrm{d}\theta_i, \Omega_i \rangle_{\mathrm{alg}}$, a *rank*-1 *Bäcklund transformation* relating them is a 6-manifold

$$\iota : N^6 \hookrightarrow M_1 \times M_2$$

with the natural projections $\pi_i : N \to M_i$ satisfying:

(1) Both π_1 and π_2 are submersions, and $\ker(\mathrm{d}\pi_1) \cap \ker(\mathrm{d}\pi_2) = 0$;

[1] Here $\langle \cdots \rangle_{\mathrm{alg}}$ denotes the ideal in $\Omega^*(M)$ generated algebraically by the differential forms enclosed in the brackets. Without the subscript 'alg', $\langle \cdots \rangle$ denotes the algebraic ideal generated by the differential forms in the bracket and their exterior derivatives.

[2] The notion of a *characteristic system* applies to hyperbolic exterior differential systems in general. See [BGH95].

[3] The reader may compare this definition, which suits the current work, with the more general version presented in [Hu19].

(2) $[\![\pi_1^*\mathrm{d}\theta_1, \pi_2^*\mathrm{d}\theta_2]\!] \equiv [\![\pi_1^*\Omega_1, \pi_2^*\Omega_2]\!] \mod \pi_1^*\theta_1, \pi_2^*\theta_2$.

Here, $[\![\pi_1^*\mathrm{d}\theta_1, \pi_2^*\mathrm{d}\theta_2]\!]$ denotes the subbundle of $\Lambda^2(T^*N)$ generated by $\pi_1^*\mathrm{d}\theta_1$ and $\pi_2^*\mathrm{d}\theta_2$. (The notation $[\![\cdots]\!]$ will be used in a similar way below.) Condition (1) implies that, given any integral surface S of $(M_1, \mathcal{I}_1)$, $\pi_1^{-1}S$ is 3-dimensional. Condition (2) implies that, on N,

$$\pi_1^*[\![\mathrm{d}\theta_1, \Omega_1]\!] \equiv \pi_2^*[\![\mathrm{d}\theta_2, \Omega_2]\!] \mod \pi_1^*\theta_1, \pi_2^*\theta_2.$$

It follows that, on $\pi_1^{-1}S$, $\pi_2^*\mathcal{I}_2$ restricts to be algebraically generated by the single 1-form $\pi_2^*\theta_2$, so the Frobenius theorem applies. In other words, $\pi_1^{-1}S$ is foliated by a 1-parameter family of surfaces whose projections via π_2 are integral surfaces of $(M_2, \mathcal{I}_2)$. The same argument works when one starts with an integral surface of $(M_2, \mathcal{I}_2)$.

1.3. **Obstructions to Existence.** Fix two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$, together with a choice of 0-adapted coframings $(\theta, \omega^1, \cdots, \omega^4)$ and $(\bar\theta, \bar\omega^1, \cdots, \bar\omega^4)$ defined on domains $V \subset M$ and $\bar{V} \subset \bar{M}$, respectively. In principle, the problem of finding a rank-1 Bäcklund transformation relating solutions of $(M, \mathcal{I})$ with those of $(\bar{M}, \bar{\mathcal{I}})$ is a problem of integration. In fact, suppose that $N^6 \subset V \times \bar{V}$ is a rank-1 Bäcklund transformation. The condition (2) in Definition 1.3 implies that, on N,

$$[\![\omega^1 \wedge \omega^2,\ \omega^3 \wedge \omega^4]\!] \equiv [\![\bar\omega^1 \wedge \bar\omega^2,\ \bar\omega^3 \wedge \bar\omega^4]\!] \mod \theta, \bar\theta,$$

where pull-back symbols are dropped for clarity.

One can always switch the pairs (ω^1, ω^2) and (ω^3, ω^4), if needed, to arrange that

$$[\![\omega^1 \wedge \omega^2]\!] \equiv [\![\bar\omega^1 \wedge \bar\omega^2]\!] \mod \theta, \bar\theta,$$
$$[\![\omega^3 \wedge \omega^4]\!] \equiv [\![\bar\omega^3 \wedge \bar\omega^4]\!] \mod \theta, \bar\theta.$$

It follows that, on N, there exist functions s^i, t^i, u^i_j $(i, j = 1, \ldots, 4)$ such that

$$\bar\omega^i = s^i\theta + t^i\bar\theta + u^i_j\omega^j$$

with $u^\alpha_r = u^s_\beta = 0$, $\det(u^\alpha_\beta), \det(u^r_s) \neq 0$ $(\alpha, \beta = 1, 2; r, s = 3, 4)$. Conversely, the existence of a rank-1 Bäcklund transformation reduces to analyzing the integrability of a Pfaffian system $(P, \mathcal{J})$, where P is the product $M \times \bar{M} \times U$ with U being a domain for the parameters s^i, t^i and u^i_j; $\mathcal{J}$ is the differential ideal generated by the four 1-forms

$$\bar\omega^i - (s^i\theta + t^i\bar\theta + u^i_j\omega^j), \quad i = 1, \ldots, 4.$$

Theoretically, this is a type of problem that can be handled by the Cartan-Kähler theory (see [BCG+13]). However, direct application of the idea above seems to have limited value for two reasons. One, 'fixing two hyperbolic Monge-Ampère systems' is not general enough; and the choice of 0-adapted coframings is quite arbitrary. Two, the calculation involved in analyzing the Pfaffian system $(P, \mathcal{J})$ often quickly becomes enormous and difficult to manage.

On the other hand, we can still analyze the existence of rank-1 Bäcklund transformations as an integrability problem. Here we take an analogous but different approach than the one described above. Instead of considering a specific pair of hyperbolic Monge-Ampère systems with chosen adapted coframings, we consider the respective G-structure bundles, say, $\mathcal{G}$ and $\bar{\mathcal{G}}$ associated to the Monge-Ampère systems. We then establish a rank-4 Pfaffian system $\mathcal{J}'$ on $\mathcal{G}\times\bar{\mathcal{G}}\times U'$, where U' is a parameter space. Suitable integral manifolds of $\mathcal{J}'$ correspond to the desired Bäcklund transformations.

This approach, based on the G-structure bundles instead of the manifolds of hyperbolic Monge-Ampère systems, allows one to work with all hyperbolic Monge-Ampère systems at the same time. By not specifying a coframing, it is possible to express integrability conditions in terms of the invariants of the Monge-Ampère systems, leading to new 'obstruction-to-existence' results.

This addresses the 'generality issue' mentioned above. Yet the magnitude of the calculation remains a challenge. Being aware of this, we have assumed, for a significant portion of this work, that the hyperbolic Monge-Ampère systems under consideration are both Euler-Lagrange (Section 4), which is the case for many known examples of Bäcklund transformations. Furthermore, at a certain stage, we assume that the rank-1 Bäcklund transformations are of a particular type, which we call 'special' (Section 5). Such Bäcklund transformations can be divided into 4 subtypes, which we name as **Type I, IIa, IIb** and **III**. Under these assumptions, we obtain our main obstruction results:

Proposition 5.2. (**A**) *If a pair of hyperbolic Euler-Lagrange systems are related by a Type I special Bäcklund transformation, then one of them must be positive, the other negative.*
(**B**) *Two hyperbolic Euler-Lagrange systems related by a Type III special Bäcklund transformation cannot be both degenerate.*

Theorem 5.1. *If two hyperbolic Euler-Lagrange systems are related by a Type IIa special rank-1 Bäcklund transformation, then each of them corresponds (up to contact equivalence) to a second order PDE of the form* $z_{xy} = F(x, y, z, z_x, z_y)$.

Theorem 5.2. *Let* $(M, \mathcal{I})$ *and* $(\bar{M}, \bar{\mathcal{I}})$ *be two hyperbolic Euler-Lagrange systems. If* $\phi : N \to M \times \bar{M}$ *defines a Type IIb special rank-1 Bäcklund transformation relating* $(M, \mathcal{I})$ *and* $(\bar{M}, \bar{\mathcal{I}})$*, then each of* $(M, \mathcal{I})$ *and* $(\bar{M}, \bar{\mathcal{I}})$ *must have a characteristic system that contains a rank-1 integrable subsystem.*

In Section 6, we discover examples of Type III special rank-1 Bäcklund transformations relating a degenerate hyperbolic Euler-Lagrange system with a non-degenerate one.

We provide a list of open questions in Section 8.

2. First Monge-Ampère Invariants

Let $(M,\mathcal{I})$ be a hyperbolic Monge-Ampère system. Let $\mathcal{G}_0$ be the principal bundle over M consisting of 0-adapted coframes of $(M,\mathcal{I})$. It is easy to verify that $\mathcal{G}_0$ is a G_0-structure, where $G_0 \subset \mathrm{GL}(5,\mathbb{R})$ is a 13-dimensional Lie subgroup. In [BGG03], the reduction of $\mathcal{G}_0$ to a G_1-structure $\mathcal{G}_1$ is performed such that the following structure equations hold on $\mathcal{G}_1$:

$$\text{(4)}\quad \mathrm{d}\begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} = -\begin{pmatrix}\phi_0 & 0 & 0 & 0 & 0\\ 0 & \phi_1 & \phi_2 & 0 & 0\\ 0 & \phi_3 & \phi_4 & 0 & 0\\ 0 & 0 & 0 & \phi_5 & \phi_6\\ 0 & 0 & 0 & \phi_7 & \phi_8\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} + \begin{pmatrix}\omega^1\wedge\omega^2 + \omega^3\wedge\omega^4\\ (V_1+V_5)\omega^0\wedge\omega^3 + (V_2+V_6)\omega^0\wedge\omega^4\\ (V_3+V_7)\omega^0\wedge\omega^3 + (V_4+V_8)\omega^0\wedge\omega^4\\ (V_8-V_4)\omega^0\wedge\omega^1 + (V_2-V_6)\omega^0\wedge\omega^2\\ (V_3-V_7)\omega^0\wedge\omega^1 + (V_5-V_1)\omega^0\wedge\omega^2\end{pmatrix},$$

where $\phi_0 = \phi_1+\phi_4 = \phi_5+\phi_8$, and $G_1 \subset G_0$ is the subgroup generated by

$$\text{(5)}\qquad g = \begin{pmatrix} a & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & A & 0\\ \mathbf{0} & 0 & B\end{pmatrix},\ A, B \in \mathrm{GL}(2,\mathbb{R}),\ a = \det(A) = \det(B),$$

and

$$\text{(6)}\qquad J = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 0 & I_2\\ \mathbf{0} & I_2 & 0\end{pmatrix} \in \mathrm{GL}(5,\mathbb{R}).$$

Definition 2.1. Let $(M,\mathcal{I})$ be a hyperbolic Monge-Ampère system. A 1-*adapted coframing* of $(M,\mathcal{I})$ with domain $U \subset M$ is a section $\boldsymbol{\eta} : U \to \mathcal{G}_1$.

Following [BGG03], we introduce the notation[4]

$$\text{(7)}\qquad S_1 := \begin{pmatrix} V_1 & V_2\\ V_3 & V_4\end{pmatrix},\quad S_2 := \begin{pmatrix} V_5 & V_6\\ V_7 & V_8\end{pmatrix}.$$

It is shown in [BGG03] that

Proposition 2.1. *Along each fiber of* $\mathcal{G}_1$,

$$\text{(8)}\qquad S_i(u\cdot g) = aA^{-1}S_i(u)B, \quad (i=1,2)$$

for any $g = \mathrm{diag}(a;A;B)$ *in the identity component of* G_1. *Moreover,*

$$\text{(9)}\qquad S_1(u\cdot J) = \begin{pmatrix} -V_4 & V_2\\ V_3 & -V_1\end{pmatrix},\quad S_2(u\cdot J) = \begin{pmatrix} V_8 & -V_6\\ -V_7 & V_5\end{pmatrix}.$$

[4] To be precise, these S_i are 1/2 times those defined in [BGG03] with the same notation.

Proposition 2.1 has a simple interpretation: the matrices S_1 and S_2 correspond to two invariant tensors under the G_1-action. More explicitly, one can verify that the quadratic form

$$\Sigma_1 := V_3\ \omega^1\omega^3 - V_1\ \omega^1\omega^4 + V_4\ \omega^2\omega^3 - V_2\ \omega^2\omega^4 \tag{10}$$

and the 2-form

$$\Sigma_2 := V_7\ \omega^1 \wedge \omega^3 - V_5\ \omega^2 \wedge \omega^3 + V_8\ \omega^1 \wedge \omega^4 - V_6\ \omega^2 \wedge \omega^4 \tag{11}$$

are G_1-invariant, which implies that Σ_1, Σ_2 are locally well-defined on M.

An infinitesimal version of Proposition 2.1 will be useful: for $i = 1, 2$,

$$\mathrm{d}S_i \equiv \begin{pmatrix} \phi_4 & -\phi_2 \\ -\phi_3 & \phi_1 \end{pmatrix} S_i + S_i \begin{pmatrix} \phi_5 & \phi_6 \\ \phi_7 & \phi_8 \end{pmatrix} \quad \mathrm{mod}\ \omega^0, \omega^1, \ldots, \omega^4. \tag{12}$$

An *Euler-Lagrange system*, in the classical calculus of variations, is a system of PDEs whose solutions correspond to the stationary points of a given first-order functional. In [BGG03], an Euler-Lagrange system is formulated as a Monge-Ampère system[5]; moreover, it is shown:

Proposition 2.2. ([BGG03]) *A hyperbolic Monge-Ampère system is locally equivalent to an Euler-Lagrange system if and only if S_2 vanishes.*

Remark 1. Proposition 2.2 says that the property of being *Euler-Lagrange* is intrinsic. From now on, we will treat this Proposition as our 'definition' of hyperbolic Euler-Lagrange (Monge-Ampère) systems.

Proposition 2.3. ([BGG03]) *A hyperbolic Monge-Ampère system corresponds to the wave equation $z_{xy} = 0$ (up to contact equivalence) if and only if $S_1 = S_2 = \mathbf{0}$.*

The following result will also be useful.

Proposition 2.4. *A hyperbolic Monge-Ampère system $(M, \mathcal{I})$, where $\mathcal{I}$ is algebraically generated by θ, $\omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$, locally corresponds to a PDE of the form $z_{xy} = F(x, y, z, z_x, z_y)$ (up to contact equivalence) if and only if each of the characteristic systems $\mathcal{I}_{10} = \langle \theta, \omega^1, \omega^2 \rangle$ and $\mathcal{I}_{01} = \langle \theta, \omega^3, \omega^4 \rangle$ admits a rank-1 integrable subsystem.*

Proof. One direction is immediate. In fact, formulating the PDE $z_{xy} = F(x, y, z, z_x, z_y)$ as a hyperbolic Monge-Ampère system, one easily notices that $\mathrm{d}x$ and $\mathrm{d}y$, respectively belonging to the two characteristic systems, are integrable.

For the other direction, assume that $(M, \mathcal{I})$ has the property that each of $\mathcal{I}_{10}$ and $\mathcal{I}_{01}$ has a rank-1 integrable subsystem; and let $(\theta, \omega^1, \ldots, \omega^4)$ be a coframing defined on a domain $U \subset M$ satisfying

$$\mathrm{d}\theta \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \quad \mathrm{mod}\ \theta.$$

For $\mathcal{I}_{10}$, this means that a certain linear combination $A\theta + B\omega^1 + C\omega^2$, where A, B, C (not all zero) are functions on U, is closed; hence, by shrinking U,

[5]See Definitions 1.3 and 1.4 of [BGG03]

if needed, we can find a function x defined on U so that $\mathrm{d}x = A\theta + B\omega^1 + C\omega^2$. Since θ is a contact form, B, C cannot both be zero. Without loss of generality, assume that $B \neq 0$. Let $\hat{\omega}^1 = A\theta + B\omega^1 + C\omega^2 = \mathrm{d}x$ and $\hat{\omega}^2 = (1/B)\omega^2$. Similarly, there exist functions A', B', C' (assuming $B' \neq 0$) and y such that $\hat{\omega}^3 = A'\theta + B'\omega^3 + C'\omega^4 = \mathrm{d}y$. Let $\hat{\omega}^4 = (1/B')\omega^4$.

Now we have

$$\mathrm{d}\theta \equiv \mathrm{d}x \wedge \hat{\omega}^2 + \mathrm{d}y \wedge \hat{\omega}^4 \mod \theta;$$

hence, the system $\langle \theta, \mathrm{d}x, \mathrm{d}y \rangle$ is completely integrable. By the Frobenius theorem, there exists a function z such that $\langle \theta, \mathrm{d}x, \mathrm{d}y \rangle = \langle dz, \mathrm{d}x, \mathrm{d}y \rangle$. In other words, there exist functions g, p, q ($g \neq 0$) defined on U such that

$$(1/g)\theta = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y.$$

This implies that

$$\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q \equiv (1/g)(\mathrm{d}x \wedge \hat{\omega}^2 + \mathrm{d}y \wedge \hat{\omega}^4) \mod \theta.$$

By Cartan's Lemma, there exists a function F such that

$$\hat{\omega}^2 \equiv g\mathrm{d}p - gF\mathrm{d}y \mod \mathrm{d}x, \theta; \qquad \hat{\omega}^4 \equiv g\mathrm{d}q - gF\mathrm{d}x \mod \mathrm{d}y, \theta.$$

The vanishing of θ and $\hat{\omega}^1 \wedge \hat{\omega}^2$ on integral surfaces then implies that locally the corresponding Monge-Ampère equation is equivalent to $z_{xy} = F(x, y, z, z_x, z_y)$. □

Now we turn to hyperbolic Euler-Lagrange systems.

By Proposition 2.1, the sign of $\det(S_1)$ is independent of the choice of 1-adapted coframings. Hence, each hyperbolic Euler-Lagrange system belongs to exactly one of the following three classes.

Definition 2.2. Given a hyperbolic Euler-Lagrange system $(M, \mathcal{I})$, it is said to be

- *positive* if $\det(S_1) > 0$;
- *negative* if $\det(S_1) < 0$;
- *degenerate* if $\det(S_1) = 0$.

Example 1. The oriented orthonormal frame bundle $\mathcal{O}$ over the Euclidean space $\mathbb{E}^3$ consists of elements of the form $(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{x} \in \mathbb{E}^3$, and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an oriented orthonormal frame at $\mathbf{x}$. On $\mathcal{O}$, define the 1-forms ω^i and ω^i_j by

$$\mathrm{d}\mathbf{x} = \mathbf{e}_i\omega^i, \quad \mathrm{d}\mathbf{e}_j = \mathbf{e}_i\omega^i_j.$$

We have the standard structure equations:

$$\begin{aligned} \mathrm{d}\omega^i &= -\omega^i_j \wedge \omega^j, \\ \mathrm{d}\omega^i_j &= -\omega^i_k \wedge \omega^k_j, \end{aligned}$$

where $\omega^i_j + \omega^j_i = 0$. Consider the natural quotient

$$\begin{aligned} \pi : \mathcal{O} &\to M := \mathbb{E}^3 \times \mathbb{S}^2 \\ (\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &\mapsto (\mathbf{x}, \mathbf{e}_3). \end{aligned}$$

The differential forms $\omega^3, \mathrm{d}\omega^3, \mathrm{d}\omega^1_2 + \omega^1 \wedge \omega^2$ are annihilated by $\ker(\mathrm{d}\pi)$ and are invariant along the fibres of π. Therefore, they are defined on M. The exterior differential system $(M, \mathcal{J})$, where

$$\mathcal{J} = \langle \omega^3, \mathrm{d}\omega^3, \mathrm{d}\omega^1_2 + \omega^1 \wedge \omega^2 \rangle_{\mathrm{alg}},$$

is hyperbolic Monge-Ampère. Its integral surfaces correspond to generalized surfaces in $\mathbb{E}^3$ with Gauss curvature $K = -1$.

Now consider the change of basis

$$\omega^1 = -\eta^1 + \eta^3, \quad \omega^2 = \eta^2 + \eta^4, \quad \omega^3 = 2\eta^0,$$
$$\omega^1_3 = \eta^2 - \eta^4, \quad \omega^2_3 = \eta^1 + \eta^3.$$

In terms of the η^i $(i = 0, \ldots, 4)$, we have $\mathcal{J} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\mathrm{alg}}$. It can be verified that $\boldsymbol{\eta} := (\eta^0, \eta^1, \ldots, \eta^4)$ is a local 1-adapted coframing of $(M, \mathcal{J})$. Calculating using this coframing, we obtain $V_2 = V_3 = 1$ with all other V_i being zero. It follows that $(M, \mathcal{J})$ is a hyperbolic Euler-Lagrange system of the negative type. (For a presentation using coordinates, see Appendix C.)

Example 2. Consider a PDE of the form $z_{xy} = f(z)$. (This is called an *f-Gordon equation*.) It corresponds to a hyperbolic Euler-Lagrange system of the degenerate type, for it is easy to verify that

$$\eta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \quad \eta^1 = \mathrm{d}x, \quad \eta^2 = \mathrm{d}p - f(z)\mathrm{d}y,$$
$$\eta^3 = \mathrm{d}y, \quad \eta^4 = \mathrm{d}q - f(z)\mathrm{d}x$$

form a 1-adapted coframing of the corresponding hyperbolic Monge-Ampère system. Using this coframing, one can calculate that $V_3 = -f'(z)$ with all other V_i being identically zero.

Example 3. Consider the Lorentzian space $\mathbb{E}^{2,1}$. Let $\pi : \mathcal{F} \to \mathbb{E}^{2,1}$ be the oriented pseudo-orthonormal frame bundle (with $\mathrm{SO}(2,1)$ fibres) consisting of $(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ satisfying

$$\mathbf{x} \in \mathbb{E}^{2,1}, \quad \mathbf{e}_i \in \mathbb{R}^3,$$
$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \ (i \neq j), \quad \mathbf{e}_1 \cdot \mathbf{e}_1 = -\mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1,$$

where $\cdot$ stands for the inner product on $\mathbb{E}^{2,1}$ with the signature $(+,-,+)$.

Define the 1-forms ω^i, ω^i_j on $\mathcal{F}$ by

$$\mathrm{d}\mathbf{x} = \mathbf{e}_i \omega^i, \qquad \mathrm{d}\mathbf{e}_j = \mathbf{e}_i \omega^i_j.$$

We have the structure equations

$$\mathrm{d}\omega^i = -\omega^i_j \wedge \omega^j,$$
$$\mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j,$$

with $\omega^2_\alpha = \omega^\alpha_2$ $(\alpha = 1, 3)$ and $\omega^1_3 + \omega^3_1 = 0$. Consider the quotient

$$\pi : \mathcal{F} \to M := \mathbb{E}^{2,1} \times \mathcal{H}$$
$$(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \mapsto (\mathbf{x}, \mathbf{e}_3),$$

where $\mathcal{H}$ is the hyperboloid in $\mathbb{R}^3$ defined by $x^2 - y^2 + z^2 = 1$.

The differential forms $\omega^3, \mathrm{d}\omega^3, \omega^1 \wedge \omega^2 - \omega^1_3 \wedge \omega^2_3$ are annihilated by $\ker(\mathrm{d}\pi)$ and are invariant along the fibres of π. Thus they are defined on M. The exterior differential system $(M, \mathcal{J})$, where

$$\mathcal{J} = \langle \omega^3, \mathrm{d}\omega^3, \omega^1 \wedge \omega^2 - \omega^1_3 \wedge \omega^2_3 \rangle_{\mathrm{alg}}$$

is hyperbolic Monge-Ampère. Its integral surfaces correspond to time-like (since $\mathbf{e}_3$, being space-like, is the normal) surfaces in $\mathbb{E}^{2,1}$ with the constant Gauss curvature $K = 1$.

Under the change of basis

$$\omega^1 = -\eta^2 + \eta^4, \quad \omega^2 = -\eta^1 - \eta^3, \quad \omega^3 = -2\eta^0,$$
$$\omega^1_3 = -\eta^3 + \eta^1, \quad \omega^2_3 = \eta^4 + \eta^2,$$

one can verify that $\boldsymbol{\eta} := (\eta^0, \eta^1, \ldots, \eta^4)$ is a local 1-adapted coframing of $(M, \mathcal{J})$. Computing using this coframing, we find that $V_2 = 1$, $V_3 = -1$, all other V_i being zero. It follows that $(M, \mathcal{J})$ is a hyperbolic Euler-Lagrange system of the positive type. (For a presentation using coordinates, see Appendix C.)

Remark 2. One can verify that the hyperbolic Monge-Ampère systems occurring in Clelland's classification [Cle02] of homogeneous rank-1 Bäcklund transformations are Euler-Lagrange.

Proposition 2.5. *A hyperbolic Euler-Lagrange system of either the positive or the negative type is* not *contact equivalent to any PDE of the form* $z_{xy} = F(x, y, z, z_x, z_y)$.

Proof. Given a hyperbolic Monge-Ampère system $(M, \mathcal{I})$, if $S_2 = 0$ and S_1 nonsingular, we can compute the *derived systems*[6] (see [BCG+13]) of $I_{10} = [\![\omega^0, \omega^1, \omega^2]\!]$ using (4) and find

$$I^{(1)}_{10} = [\![\omega^1, \omega^2]\!], \qquad I^{(2)}_{10} = 0.$$

It follows that I_{10} has no integrable subsystem. A similar argument works for I_{01}. By Proposition 2.4, $(M, \mathcal{I})$ cannot be contact equivalent to a PDE of the form $z_{xy} = F(x, y, z, z_x, z_y)$. □

3. The Bäcklund-Pfaff System

In this section, we prove that, given two hyperbolic Monge-Ampère systems, the existence of a *rank*-1 *Bäcklund transformation* relating them can be formulated as the integrability of a rank-4 Pfaffian system.

We start by fixing some notations.

Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Monge-Ampère systems. Let $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$ be the respective G_1-structures (see Section 2). Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_4)$

[6]Given a Pfaffian system $I \subset T^*M$, its derived systems are defined recursively: $I^{(0)} = I$, $I^{(j+1)} = [\![\omega \in \Gamma(I^{(j)}) |\ \mathrm{d}\omega \equiv 0 \mod I^{(j)}]\!] \subset T^*M$. There always exists a $k \geqslant 0$ such that $I^{(\ell)} = I^{(k)} =: I^{(\infty)}$ for all $\ell \geqslant k$. Any integrable subsystem of I is contained in $I^{(\infty)}$.

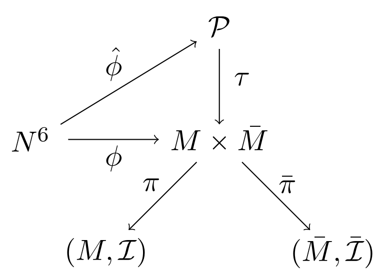
and $\beta = (\beta_0, \beta_1, \ldots, \beta_4)$ be the tautological 1-forms on $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, respectively. And let $\mathcal{P}$ be the product of $\mathcal{G}_1 \times \bar{\mathcal{G}}_1$ with a space of parameters:

$$\mathcal{P} := \left\{ (u, \bar{u}, s, t, \mu, \epsilon) \,\middle|\, \begin{array}{l} u \in \mathcal{G}_1,\ \bar{u} \in \bar{\mathcal{G}}_1;\ s, t \in \mathbb{R}^4; \\ \mu \geqslant 1;\ \epsilon = \pm 1\ (\epsilon\mu^2 \neq 1) \end{array} \right\} \subset \mathcal{G}_1 \times \bar{\mathcal{G}}_1 \times \mathbb{R}^9 \times \{\pm 1\}. \tag{13}$$

Proposition 3.1. *An embedded* 6*-manifold* $\phi : N \to M \times \bar{M}$ *is a rank-*1 *Bäcklund transformation if and only if* N *admits a lifting* $\hat{\phi} : N \to \mathcal{P}$ *that is an integral manifold of a rank-*4 *Pfaffian system* $\mathcal{J}$ *with the* independence condition $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_0 \wedge \beta_0 \neq 0$*, where* $\mathcal{J}$ *is generated by*

$$\begin{aligned} \theta_1 &:= \beta_1 - s_1\alpha_0 - t_1\beta_0 - \mu\alpha_1, \\ \theta_2 &:= \beta_2 - s_2\alpha_0 - t_2\beta_0 - \mu\alpha_2, \\ \theta_3 &:= \beta_3 - s_3\alpha_0 - t_3\beta_0 - \mu^{-1}\alpha_3, \\ \theta_4 &:= \beta_4 - s_4\alpha_0 - t_4\beta_0 - \epsilon\mu^{-1}\alpha_4, \end{aligned} \tag{14}$$

at each $(u, \bar{u}, s_1, \ldots, s_4, t_1, \ldots, t_4, \mu, \epsilon) \in \mathcal{P}$*.*



Proof. Let $\tau, \pi, \bar{\pi}$ denote the obvious projections (see the diagram above). By the construction of $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, on $\mathcal{P}$, we have

$$\begin{aligned} (\pi \circ \tau)^*(\mathcal{I}) &= \langle \alpha_0, \alpha_1 \wedge \alpha_2, \alpha_3 \wedge \alpha_4 \rangle, \\ [\![(\pi \circ \tau)^*(\theta)]\!] &\equiv [\![\alpha_0]\!], \end{aligned}$$

and similarly for $\bar{\mathcal{I}}$ and the β_i. Now assume that ϕ admits a lifting $\hat{\phi}$ that integrates the Pfaffian system $\mathcal{J}$. It is easy to see that, on N,

$$\begin{aligned} \hat{\phi}^*([\![\alpha_1 \wedge \alpha_2, \alpha_3 \wedge \alpha_4]\!]) &\equiv \hat{\phi}^*([\![\beta_1 \wedge \beta_2, \beta_3 \wedge \beta_4]\!]) \\ &\equiv \hat{\phi}^*([\![\mathrm{d}\alpha_0, \mathrm{d}\beta_0]\!]) \mod \hat{\phi}^*\alpha_0, \hat{\phi}^*\beta_0. \end{aligned}$$

In the last congruence, we have used the assumption that $\mu^2 \neq \epsilon$, which guarantees that the bundle $[\![\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4,\ \beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4]\!]$ has rank 2 modulo α_0, β_0 when pulled back to N. It follows that $\phi : N \to M \times \bar{M}$ defines a rank-1 Bäcklund transformation.

Conversely, suppose that $\phi : N \to M \times \bar{M}$ defines a rank-1 Bäcklund transformation. Let $\boldsymbol{\eta} := (\eta_0, \eta_1, \ldots, \eta_4)$ (resp. $\boldsymbol{\xi} := (\xi_0, \xi_1, \ldots, \xi_4)$) be a local 1-adapted coframing defined on a domain in M (resp. $\bar{M}$). We have, by definition,

$$\phi^*([\![\eta_1 \wedge \eta_2, \eta_3 \wedge \eta_4]\!]) \equiv \phi^*([\![\xi_1 \wedge \xi_2, \xi_3 \wedge \xi_4]\!]) \mod \phi^*\eta_0, \phi^*\xi_0.$$

By switching the pairs (η_1, η_2) and (η_3, η_4) and by shrinking N, if needed, we can assume that, on N,

$$\begin{aligned} [\![\eta_1 \wedge \eta_2]\!] &\equiv [\![\xi_1 \wedge \xi_2]\!] \mod \eta_0, \xi_0, \\ [\![\eta_3 \wedge \eta_4]\!] &\equiv [\![\xi_3 \wedge \xi_4]\!] \mod \eta_0, \xi_0, \end{aligned}$$

where the pull-back symbol is dropped for convenience. Consequently, there exist 16 functions $s_1, \ldots, s_4, t_1, \ldots, t_4, u_1, \ldots, u_8$ defined on N such that, when restricted to N,

$$\begin{aligned} \xi_1 &= s_1\eta_0 + t_1\xi_0 + u_1\eta_1 + u_2\eta_2, \\ \xi_2 &= s_2\eta_0 + t_2\xi_0 + u_3\eta_1 + u_4\eta_2, \\ \xi_3 &= s_3\eta_0 + t_3\xi_0 + u_5\eta_3 + u_6\eta_4, \\ \xi_4 &= s_4\eta_0 + t_4\eta_0 + u_7\eta_3 + u_8\eta_4. \end{aligned} \tag{15}$$

Here $u_1u_4 - u_2u_3$ and $u_5u_8 - u_7u_6$ are both nonvanishing. Moreover, since $[\![\mathrm{d}\eta_0, \mathrm{d}\xi_0]\!]$ has rank 2 modulo η_0, ξ_0, we have

$$u_1u_4 - u_2u_3 \neq u_5u_8 - u_6u_7.$$

Using the flexibility of choosing the 1-adapted coframings $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ (see (5), (6)), we can normalize some of the u_i. To be specific, we can apply $\mathrm{SL}(2, \mathbb{R})$-actions (i.e., acting on $\boldsymbol{\eta}$ pointwise by a matrix g in the form of (5) with $a = 1$) to (η_1, η_2) and (η_3, η_4) to arrange that

$$u_2 = u_3 = u_6 = u_7 = 0$$

and

$$u_4 = \pm u_1, \quad u_8 = \pm u_5 \quad (u_1, u_5 > 0).$$

Then we transform $(\eta_0, \eta_1, \ldots, \eta_4)$ pointwise by a diagonal matrix in G_1 to arrange that

$$u_1 = u_4 > 0, \quad u_5 = u_1^{-1}, \quad u_8 = \epsilon u_1^{-1} \quad (\epsilon = \pm 1).$$

Meanwhile, this transformation scales the s_i ($i = 1, \ldots, 4$). Finally, if $u_1 = u_4 < 1$, then we switch the pairs of indices $(1, 2)$ and $(3, 4)$ for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in (15) and multiply the new η_0, η_2, η_4 by ϵ; the resulting u_i will satisfy

$$u_1 = u_4 > 1, \quad u_5 = u_1^{-1}, \quad u_8 = \epsilon u_1^{-1}.$$

It follows that the $(\boldsymbol{\eta}, \boldsymbol{\xi}, (s_i), (t_i), u_1, \epsilon)$ thus constructed defines a map $\hat{\phi} : N \to \mathcal{P}$ that is a lifting of ϕ and an integral manifold of $\mathcal{J}$. □

In light of Proposition 3.1, we make the following definitions.

Definition 3.1. The system $(\mathcal{P}, \mathcal{J})$ in Proposition 3.1 is called the 0-*refined Bäcklund-Pfaff system* for rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Definition 3.2. A 6-dimensional integral manifold (satisfying the independence condition described in Proposition 3.1) of the 0-refined Bäcklund-Pfaff system $(\mathcal{P}, \mathcal{J})$ is called a 0-*refined lifting* of the underlying Bäcklund transformation.

In these terms, Proposition 3.1 says that each rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems has a 0-refined lifting. Of course, given such a Bäcklund transformation $\phi : N \to M \times \bar{M}$, its 0-refined liftings are not unique.

Lemma 3.1. *Let $\phi : N \to M \times \bar{M}$ be a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. The functions $\mu \circ \hat{\phi}$ and $\epsilon \circ \hat{\phi}$ are independent of the choice of 0-refined liftings $\hat{\phi}$ of ϕ.*

Proof. Clearly, for different choices of $\hat{\phi}$, the 1-forms $\phi^*\alpha_0$ and $\phi^*\beta_0$ only change by scaling. On N, the quotient λ_1/λ_2 between the two solutions λ_1, λ_2 of the equation

$$(\mathrm{d}\beta_0 + \lambda \mathrm{d}\alpha_0)^2 \equiv 0 \mod \alpha_0, \beta_0$$

is independent of the scaling of α_0 and β_0 but may depend on the order of the pair (λ_1, λ_2). By (14), either λ_1/λ_2 or λ_2/λ_1 must be equal to $\epsilon\mu^4$.

When $|\lambda_1/\lambda_2| = 1$, it is necessary that $\mu = 1$ and $\epsilon = -1$. When $|\lambda_1/\lambda_2| \neq 1$, the ambiguity of ordering λ_1 and λ_2 can be eliminated by requiring that $|\lambda_1/\lambda_2| > 1$, and hence $\lambda_1/\lambda_2 = \epsilon\mu^4$. It follows that μ and ϵ are both independent of the lifting. □

Remark 3. **A.** There is a simple geometric interpretation for the two possible values of ϵ. Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be as above. Suppose that $\phi : N \to M \times \bar{M}$ defines a rank-1 Bäcklund transformation. The 4-plane field $\mathcal{D} := [\![\alpha_0, \beta_0]\!]^\perp$ on N is independent of the choice of 0-refined liftings of ϕ. When restricted to $\mathcal{D}$, the 4-forms $(\mathrm{d}\alpha_0)^2$ and $(\mathrm{d}\beta_0)^2$ define two orientations. If $\epsilon = 1$, these two orientations are the same; if $\epsilon = -1$, they are distinct.

B. If, for a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems, $\epsilon = 1$, then $\mu > 1$. This is because, if $\mu = 1$, then the condition (2) in Definition 1.3 will not hold.

4. Obstructions to Integrability

In this section, we express some obstructions to the integrability of $(\mathcal{P}, \mathcal{J})$ in terms of the invariants of the two hyperbolic Monge-Ampère systems.

For convenience, we introduce new notations below.

(A) Let $\eta_1 := \alpha_1,\ \eta_2 := \alpha_2,\ \eta_3 := \alpha_3,\ \eta_4 := \alpha_4,\ \eta_5 := \alpha_0,\ \eta_6 := \beta_0$.

(B) The components of the pseudo-connection 1-form on $\mathcal{G}_1$ (see (4)) are denoted by $\varphi_0, \varphi_1, \ldots, \varphi_8$; those on $\bar{\mathcal{G}}_1$ are denoted by $\psi_0, \psi_1, \ldots, \psi_8$.

On $\mathcal{P}$, differentiating the θ_i and reducing modulo $\theta_1, \ldots, \theta_4$ yields the following congruences:

$$\mathrm{d}\theta_i \equiv -\pi_{i\alpha} \wedge \eta_\alpha + \tau_i \qquad \mod \theta_1, \ldots, \theta_4 \qquad (i = 1, \ldots, 4), \tag{16}$$

where summation over repeated indices is intended; $\pi_{i\alpha}$ $(i = 1, \ldots, 4; \alpha = 1, \ldots, 6)$ are linear combinations of the 1-forms in the set

$$\begin{aligned}\mathcal{S} :=\{&\mathrm{d}(s_1), \ldots, \mathrm{d}(s_4);\ \mathrm{d}(t_1), \ldots, \mathrm{d}(t_4);\ \mathrm{d}\mu;\\ &\varphi_0, \ldots, \varphi_3, \varphi_5, \varphi_6, \varphi_7;\ \psi_0, \ldots, \psi_3, \psi_5, \psi_6, \psi_7\};\end{aligned}$$

the components of the *torsion* $T := (\tau_i)$ are of the form

$$\tau_i = \frac{1}{2} T_{ijk} \eta_j \wedge \eta_k,$$

for some functions T_{ijk} defined on $\mathcal{P}$ satisfying $T_{ijk} = -T_{ikj}$.

We use a standard method (see [BCG$^+$13]) to obtain from (16) obstructions to the integrability of $(\mathcal{P}, \mathcal{J})$. The key is that, on an integral manifold of $(P, \mathcal{J})$, $\theta_i, \mathrm{d}\theta_i$ all vanish and $\pi_{i\alpha}$ are linear combinations of $\eta_1, \eta_2, \ldots, \eta_6$. It follows that an integral manifold of $(\mathcal{P}, \mathcal{J})$ can only exist within the locus along which τ_i can be fully absorbed by $\pi_{i\alpha} \wedge \eta_\alpha$, as long as one add suitable linear combinations of η_α to the 1-forms in $\mathcal{S}$.

This is precisely the approach we take in practice: we keep adjusting the 1-forms in $\mathcal{S}$ by adding linear combinations of η_α to them until a point when the rank of (τ^i) cannot be further reduced. The remaining (τ^i) must vanish along integral manifolds of $(P, \mathcal{J})$.

In our case, the matrix $(\pi_{i\alpha})$ (called the *tableau*) takes the form

$$(\pi_{i\alpha}) = \begin{pmatrix} \pi_1 & \pi_2 & 0 & 0 & \pi_3 & \pi_4 \\ \pi_5 & \pi_6 & 0 & 0 & \pi_7 & \pi_8 \\ 0 & 0 & \pi_9 & \pi_{10} & \pi_{11} & \pi_{12} \\ 0 & 0 & \pi_{13} & \pi_{14} & \pi_{15} & \pi_{16} \end{pmatrix},$$

where $\pi_1, \ldots, \pi_{16}$ are 1-forms[7] linearly independent of $\theta_1, \ldots, \theta_4, \eta_1, \ldots, \eta_6$ and among themselves. It is easy to see that all terms in τ_i can be absorbed except the $\eta_3 \wedge \eta_4$ terms in τ_1 and τ_2 and the $\eta_1 \wedge \eta_2$ terms in τ_3 and τ_4.

Calculation yields

$$\begin{aligned}
\mathrm{d}\theta_1 &\equiv -\frac{\mu^2 s_1 + \epsilon t_1}{\mu^2} \eta_3 \wedge \eta_4 \qquad &&\mod \theta_1, \ldots, \theta_4, \eta_1, \eta_2, \eta_5, \eta_6,\\
\mathrm{d}\theta_2 &\equiv -\frac{\mu^2 s_2 + \epsilon t_2}{\mu^2} \eta_3 \wedge \eta_4 \qquad &&\mod \theta_1, \ldots, \theta_4, \eta_1, \eta_2, \eta_5, \eta_6,\\
\mathrm{d}\theta_3 &\equiv -(\mu^2 t_3 - s_3) \eta_1 \wedge \eta_2 \qquad &&\mod \theta_1, \ldots, \theta_4, \eta_3, \eta_4, \eta_5, \eta_6,\\
\mathrm{d}\theta_4 &\equiv -(\mu^2 t_4 - s_4) \eta_1 \wedge \eta_2 \qquad &&\mod \theta_1, \ldots, \theta_4, \eta_3, \eta_4, \eta_5, \eta_6.
\end{aligned}$$

As a result, we have proved:

[7] One can think of $(\pi_{i\alpha})$ as an $\mathcal{A}$-valued 1-form, where $\mathcal{A} \subset \mathbb{R}^4 \otimes (\mathbb{R}^6)^*$ is a vector subspace. Here, the $\mathbb{R}$-valued 1-forms $\pi_1, \ldots, \pi_{16}$ are just a set of 'form coordinates' for $(\pi_{i\alpha})$.

Lemma 4.1. *Integral manifolds of $(\mathcal{P}, \mathcal{J})$ must be contained in the locus $\mathcal{L} \subset \mathcal{P}$ defined by the equations*

$$\begin{aligned} s_3 &= -t_3\mu^2, & s_4 &= -t_4\mu^2, \\ t_1 &= -\epsilon s_1\mu^2, & t_2 &= -\epsilon s_2\mu^2. \end{aligned}$$

Definition 4.1. Let $\mathcal{P}_1 \subset \mathcal{P}$ be the locus defined by the equations

$$s_1 = t_1 = s_3 = t_3 = 0, \quad s_4 = -t_4\mu^2, \quad t_2 = -\epsilon s_2\mu^2.$$

Let $\mathcal{J}_1$ be the pull-back of $\mathcal{J}$ to $\mathcal{P}_1$. The rank-4 Pfaffian system $(\mathcal{P}_1, \mathcal{J}_1)$ is called the 1-*refined Bäcklund-Pfaff system* for rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Definition 4.2. A 6-dimensional integral manifold of the 1-refined Bäcklund-Pfaff system $(\mathcal{P}_1, \mathcal{J}_1)$ is called a 1-*refined lifting* of the underlying Bäcklund transformation.

Proposition 4.1. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be as above. Any rank-*1 *Bäcklund transformation $\phi : N \to M \times \bar{M}$ admits a* 1*-refined lifting.*

Proof. By the previous discussion, there exists a 0-refined lifting $\hat{\phi}$ of ϕ such that, when pulled-back via $\hat{\phi}$ to N,

$$\begin{aligned} \beta_1 &= s_1\alpha_0 - \epsilon s_1\mu^2\beta_0 + \mu\alpha_1, \\ \beta_2 &= s_2\alpha_0 - \epsilon s_2\mu^2\beta_0 + \mu\alpha_2, \\ \beta_3 &= -t_3\mu^2\alpha_0 + t_3\beta_0 + \mu^{-1}\alpha_3, \\ \beta_4 &= -t_4\mu^2\alpha_0 + t_4\beta_0 + \epsilon\mu^{-1}\alpha_4. \end{aligned}$$

If $s_2 \neq 0$, we transform the pairs $(\alpha_1, \alpha_2)^T$ and $(\beta_1, \beta_2)^T$ simultaneously by

$$h_1 = \begin{pmatrix} 1 & -s_1/s_2 \\ 0 & 1 \end{pmatrix}.$$

If $s_2 = 0$ but $s_1 \neq 0$, we transform the pairs $(\alpha_1, \alpha_2)^T$ and $(\beta_1, \beta_2)^T$ simultaneously by

$$h_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

such that the previous case applies. These transformations correspond to choosing new 0-refined liftings, and the result is a 0-refined lifting with $s_1 = 0$. In a similar way, we can choose 0-refined liftings such that, in addition, $t_3 = 0$. This completes the proof. □

Now $\mathcal{J}_1$ is generated by

$$\begin{aligned} \theta_1 &= \beta_1 - \mu\alpha_1, \\ \theta_2 &= \beta_2 - s_2\alpha_0 + \epsilon s_2\mu^2\beta_0 - \mu\alpha_2, \\ \theta_3 &= \beta_3 - \mu^{-1}\alpha_3, \\ \theta_4 &= \beta_4 + t_4\mu^2\alpha_0 - t_4\beta_0 - \epsilon\mu^{-1}\alpha_4. \end{aligned} \tag{17}$$

Given a rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$, it is easy to see that whether the product s_2t_4 locally vanishes is independent of the choice of 1-refined liftings of ϕ. It turns out that the case when $s_2t_4 \equiv 0$ is quite restrictive when both $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ are Euler-Lagrange systems.

Proposition 4.2. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Euler-Lagrange systems. If there exists a rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ such that $s_2t_4 = 0$ on a 1-refined lifting of ϕ, then both $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ must be contact equivalent to the system representing the wave equation*

$$z_{xy} = 0.$$

Proof. By the Euler-Lagrange assumption and Proposition 2.2, we have $S_2 = \bar{S}_2 = \mathbf{0}$. Let

$$S_1 = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}, \qquad \bar{S}_1 = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}.$$

By Proposition 2.3, it suffices to show that, on any integral manifold of the 1-refined Bäcklund-Pfaff system $(\mathcal{P}_1, \mathcal{J}_1)$, we have $S_1 = \bar{S}_1 = \mathbf{0}$.

First, we assume that $s_2 = 0$. Restricted to the locus defined by $s_2 = 0$ in $\mathcal{P}_1$, the tableau $(\pi_{i\alpha})$ associated to $(\mathcal{P}_1, \mathcal{J}_1)$ satisfies

$$\pi_{i\alpha} = 0, \quad (i = 1, 2;\ \alpha = 3, \ldots, 6).$$

As a result, for $i, j = 3, \ldots, 6$ and $i \neq j$, the $\eta^i \wedge \eta^j$ terms in $\mathrm{d}\theta_1$ and $\mathrm{d}\theta_2$ cannot be absorbed, and the corresponding coefficients must vanish on any integral manifold of $(\mathcal{P}_1, \mathcal{J}_1)$. Calculating with Maple$^{\mathrm{TM}}$, we find that

$$\mathrm{d}\theta_1 \equiv \mu(V_1\eta_3 + V_2\eta_4) \wedge \eta_5 - \frac{1}{\mu}(W_1\eta_3 + \epsilon W_2\eta_4) \wedge \eta_6 + W_2t_4\mu^2\eta_5 \wedge \eta_6,$$
$$\mathrm{d}\theta_2 \equiv \mu(V_3\eta_3 + V_4\eta_4) \wedge \eta_5 - \frac{1}{\mu}(W_3\eta_3 + \epsilon W_4\eta_4) \wedge \eta_6 + W_4t_4\mu^2\eta_5 \wedge \eta_6,$$

both congruences being reduced modulo $\theta_1, \ldots, \theta_4, \eta_1, \eta_2$. It follows that $S_1 = \bar{S}_1 = \mathbf{0}$.

The case when $t_4 = 0$ is similar. □

Proposition 4.3. *On any 1-refined lifting of a rank-1 Bäcklund transformation relating two hyperbolic Euler-Lagrange systems, the following expressions must vanish:*

$$\Phi_1 := -\mu^4V_1 + \epsilon W_1, \qquad \Phi_2 := -\mu^4V_2 + W_2,$$
$$\Phi_3 := \mu^4W_4 - V_4, \qquad \Phi_4 := \mu^4W_2 - V_2.$$

Proof. This is evident when such a Bäcklund transformations satisfies $s_2t_4 = 0$. In fact, by Proposition 4.2, the functions V_i and W_i vanish identically on $\mathcal{P}_1$. Otherwise, restricting to the open subset of $\mathcal{P}_1$ defined by $s_2t_4 \neq 0$, a calculation similar to that in the proof of Proposition 4.2 shows that the torsion of the Pfaffian system $\mathcal{J}_1$ can be absorbed only when Φ_i $(i = 1, \ldots, 4)$ are all zero. □

Corollary 4.2. *If two hyperbolic Euler-Lagrange systems are related by a rank-1 Bäcklund transformation with $\epsilon = 1$, then they are either both degenerate or both nondegenerate.*

Proof. We have noted above (see Remark 3) that, if a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems satisfies $\epsilon = 1$, then $\mu > 1$. The vanishing of Φ_2 and Φ_4 on a 1-refined lifting of such a Bäcklund transformation then implies that, on such a lifting,

$$V_2 = W_2 = 0.$$

By the vanishing of Φ_1 and Φ_3, it is easy to see that the matrices S_1 and $\bar{S}_1$ are either both degenerate or both nondegenerate. □

Note that, in the proof of Corollary 4.2, the condition $V_2 = W_2 = 0$ is meaningful only if it is independent of the choice of 1-refined liftings of a Bäcklund transformation. To make this point explicit, we prove the following proposition, which shows how 1-refined liftings of a rank-1 Bäcklund transformation relate to each other.

Proposition 4.4. *Let ϕ be a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems satisfying $s_2t_4 \neq 0$ on its 1-refined liftings. There exists a subgroup $H \subset G_1 \times G_1$ such that any two 1-refined liftings of ϕ are related, in the $\mathcal{G}_1 \times \bar{\mathcal{G}}_1$ component, by an H-valued transformation. Moreover,*

- **i.** *If $\mu > 1$, then H is the subgroup generated by elements of the form $h = (g, g')$ where*

$$g = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 & 0 \\ 0 & h_3 & h_0h_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_4 & h_0h_2^{-1} \end{pmatrix} \in G_1, \tag{18}$$

  and g' is the result of replacing h_4 in g by ϵh_4.
- **ii.** *If $\mu = 1$, then H is generated by the subgroup in case* **i** *and the element*

$$h_J = (J', J),$$

  where J is as in (6) and $J' = \mathrm{diag}(-1, 1, -1, 1, -1)J$.

Proof. By (17), any two 1-refined liftings of ϕ must be related by a pointwise H-action, where $H \subset G_1 \times G_1$ is a subgroup.

Note that G_1 has two connected components: G_e consisting of those elements of the form (5) and $G_1 \backslash G_e = JG_e$.

When $\mu > 1$, one must have $H \subset G_e \times G_e$. This is because switching $[\![\alpha_1, \alpha_2]\!]$ with $[\![\alpha_3, \alpha_4]\!]$ enforces switching $[\![\beta_1, \beta_2]\!]$ with $[\![\beta_3, \beta_4]\!]$, which transforms μ into μ^{-1}, which is not permissible. Now suppose that $h = (g, g') \in H$. It is easy to see, by (17) and the assumption $s_2t_4 \neq 0$ that

$g \in H$ must be of the form (18). The form of g' is then determined, as stated in **i**.

When $\mu = 1$, we have $\epsilon = -1$. In this case, we do allow switching between the indices $(1,2)$ and $(3,4)$, by an action of J on the β's and a corresponding action on the α's so that the form of (17) is preserved.

Finally, note that a pointwise action by H described above transforms a 1-refined lifting of ϕ to a 1-refined lifting of ϕ.

This completes the proof. □

Remark 4. By (8) and (9), $h = (g, g')$ and $h_J = (J', J)$ in Proposition 4.4 act on $S_1 = (V_i)$ and $\bar{S}_1 = (W_i)$ in the following way:

$$S_1(u \cdot g) = h_0 \begin{pmatrix} h_1^{-1} & 0 \\ -h_0^{-1}h_3 & h_0^{-1}h_1 \end{pmatrix} S_1(u) \begin{pmatrix} h_2 & 0 \\ h_4 & h_0 h_2^{-1} \end{pmatrix},$$

$$\bar{S}_1(\bar{u} \cdot g') = h_0 \begin{pmatrix} h_1^{-1} & 0 \\ -h_0^{-1}h_3 & h_0^{-1}h_1 \end{pmatrix} \bar{S}_1(\bar{u}) \begin{pmatrix} h_2 & 0 \\ \epsilon h_4 & h_0 h_2^{-1} \end{pmatrix},$$

$$S_1(u \cdot J') = \begin{pmatrix} V_4 & V_2 \\ V_3 & V_1 \end{pmatrix}(u), \quad \bar{S}_1(\bar{u} \cdot J) = \begin{pmatrix} -W_4 & W_2 \\ W_3 & -W_1 \end{pmatrix}(\bar{u}).$$

In particular, if $W_2 = V_2 = 0$ holds on a 1-refined lifting of ϕ, then it holds on any other 1-refined lifting of ϕ.

5. A Special Class of Bäcklund Transformations

In the previous section, we have seen that, to a rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$, we can associate a function $\mu : N \to [1, \infty)$ that is independent[8] of the 0-refined liftings of ϕ.

Definition 5.1. A rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems is said to be *special* if $\mu = 1$.

For the rest of this section, we will focus on special Bäcklund transformations relating two hyperbolic Euler-Lagrange systems. A motivation for this is that many classical Bäcklund transformations are of this type (cf. [Cle02], [Cle18]).

By Proposition 4.3, given a special rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ relating two hyperbolic Euler-Lagrange systems, the following equalities must hold on any 1-refined lifting of ϕ:

$$\epsilon = -1, \quad W_1 = -V_1, \quad W_2 = V_2, \quad W_4 = V_4. \tag{19}$$

By Remark 4, these conditions are invariant under the H-action defined in Proposition 4.4.

Now let $\mathcal{P}_s \subset \mathcal{P}_1$ be defined by the equations

$$\mu = 1, \qquad \epsilon = -1.$$

[8]In fact, it is easy to see, from the point of view of [Hu19] that μ is a local invariant of the Bäcklund transformation.

Proposition 5.1. *Any 1-refined lifting of a special rank-1 Bäcklund transformation relating two hyperbolic Euler-Lagrange systems is completely contained in $\mathcal{P}_s$. Moreover, on such a lifting, in addition to (19), we have*

$$V_3 + W_3 + 2s_2t_4 = 0. \tag{20}$$

Proof. Restricting to $\mathcal{P}_s$, the generators θ_i of $\mathcal{J}_1$ satisfy congruences of the form:

$$\mathrm{d}\theta_i \equiv -\pi_{i\alpha}\wedge\eta_\alpha + \tau_i, \quad \mod \theta_1,\dots,\theta_4 \quad (i=1,\dots,4).$$

The tableau $(\pi_{i\alpha})$ now takes the form

$$(\pi_{i\alpha}) = \begin{pmatrix} -\pi_1 & -\pi_2 & 0 & 0 & s_2\pi_8 & s_2\pi_8 \\ -\pi_3 & \pi_1-\pi_7 & 0 & 0 & -s_2\pi_7-\pi_9 & -\pi_9 \\ 0 & 0 & -\pi_4 & -\pi_5 & -t_4\pi_{10} & t_4\pi_{10} \\ 0 & 0 & -\pi_6 & \pi_7-\pi_4 & t_4\pi_7+\pi_{11} & -\pi_{11} \end{pmatrix},$$

where, reduced modulo $\eta_0,\eta_1,\dots,\eta_4$,

$$\begin{aligned} &\pi_1 \equiv \varphi_1-\psi_1, && \pi_4\equiv\varphi_5-\psi_5, && \pi_7\equiv\varphi_0-\psi_0, && \pi_{10}\equiv\psi_6 \\ &\pi_2\equiv\varphi_2-\psi_2, && \pi_5\equiv\varphi_6+\psi_6, && \pi_8\equiv\psi_2, && \pi_{11}\equiv t_4\psi_5-\mathrm{d}(t_4). \\ &\pi_3\equiv\varphi_3-\psi_3, && \pi_6\equiv-\varphi_7-\psi_7, && \pi_9\equiv s_2\psi_1-\mathrm{d}(s_2), && \end{aligned}$$

By a calculation using Maple™, it is easy to see that the torsion τ_i can be absorbed only if the equations (19) and (20) hold. □

The equalities (19) and (20) tell us which Euler-Lagrange systems may be special Bäcklund-related. In particular, by Propositions 4.4 and 2.1, it is easy to see that whether V_2 (hence W_2) vanishes is independent of the choice of 1-refined liftings. Thus, we may locally[9] classify special rank-1 Bäcklund transformations relating two hyperbolic Euler-Lagrange systems into the following three types:

Type I. $V_2 = 0$, $\det(S_1) = V_1V_4 \neq 0$;
Type II. $V_2 = 0$, $\det(S_1) = V_1V_4 = 0$;
Type III. $V_2 \neq 0$.

Proposition 5.2. (**A**) *If a pair of hyperbolic Euler-Lagrange systems are related by a Type I special Bäcklund transformation, then one of them must be positive, the other negative.*
(**B**) *Two hyperbolic Euler-Lagrange systems related by a Type III special Bäcklund transformation cannot be both degenerate.*

Proof. For Part (**A**), it is immediate by (19) that, on a 1-refined lifting, $\det(S_1) = V_1V_4 = -W_1W_4 = -\det(\bar{S}_1) \neq 0$. Therefore, one of the Euler-Lagrange systems being Bäcklund-related is positive, the other negative.

To prove Part (**B**), first apply Proposition 4.4 to show that, in this case, one can always find a 1-refined lifting on which $V_1 = V_4 = 0$. (By Remark 4,

[9]Namely, the conditions below hold on an entire open subset of N.

this can be achieved by acting on an initial 1-refined lifting by an element $h \in H$ with $h_0 = h_1 = h_2 = 1$, $h_3 = V_4/V_2$, $h_4 = -V_1/V_2$.) For such a 1-refined lifting, by (20), the two Euler-Lagrange systems can be both degenerate only when $s_2t_4 = 0$. By Proposition 4.2, both S_1 and $\bar{S}_1$ must vanish, which impossible since we have assumed $V_2 \neq 0$. □

Now we focus on Type II.

Let $\phi : N \to M \times \bar{M}$ be a Type II special rank-1 Bäcklund transformation in the sense above. By Propositions 2.1 and 4.4, whether the pair (V_1, V_4) vanishes is independent of the choice of 1-refined liftings of ϕ. It follows that ϕ must be one of the following two types.

Type IIa: on any 1-refined lifting of ϕ, $(V_1, V_4) = 0$;
Type IIb: on any 1-refined lifting of ϕ, $(V_1, V_4) \neq 0$.

5.1. **Type IIa.** In this case, (19) implies that

$$W_1 = W_2 = W_4 = 0.$$

If locally either V_3 or W_3 is zero, which is independent of the choice of 1-refined liftings, the underlying Bäcklund transformation must relate a hyperbolic Euler-Lagrange system with the system corresponding to $z_{xy} = 0$. See [CI09] and [Zvy91] for a classification of all hyperbolic Monge-Ampère systems that are rank-1 Bäcklund-related to the equation $z_{xy} = 0$.

More generally, we have the following theorem.

Theorem 5.1. *If two hyperbolic Euler-Lagrange systems are related by a Type IIa special rank-*1 *Bäcklund transformation, then each of them corresponds (up to contact equivalence) to a second order PDE of the form*

$$z_{xy} = F(x, y, z, z_x, z_y).$$

Proof. By definition, any Type IIa special Bäcklund transformation admits a 1-refined lifting that is completely contained in the locus $\mathcal{P}_{\mathrm{IIa}} \subset \mathcal{P}_1$ defined by the equations

$$\mu = 1, \quad \epsilon = -1, \quad V_1 = V_2 = V_4 = W_1 = W_2 = W_4 = 0.$$

Let $\mathcal{G}_2 \subset \mathcal{G}_1$ be the subbundle defined by $V_1 = V_2 = V_4 = 0$; similarly, let $\bar{\mathcal{G}}_2 \subset \bar{\mathcal{G}}_1$ be the subbundle defined by $W_1 = W_2 = W_4 = 0$. It is clear that $\mathcal{P}_{\mathrm{IIa}}$ is the product of $\mathcal{G}_2$, $\bar{\mathcal{G}}_2$, and a space of parameters with coordinates (s_2, t_4).

The calculations below are performed using Maple$^{\mathrm{TM}}$.

By (12), on $\mathcal{G}_2$, there exist functions P_{ij} and V_{3j} such that

$$\begin{aligned}
\varphi_2 &= P_{20}\alpha_0 + P_{21}\alpha_1 + \cdots + P_{24}\alpha_4, \\
\varphi_6 &= P_{60}\alpha_0 + P_{61}\alpha_1 + \cdots + P_{64}\alpha_4, \\
\mathrm{d}(V_3) &= (\varphi_1 + \varphi_5)V_3 + V_{30}\alpha_0 + V_{31}\alpha_1 + \cdots + V_{34}\alpha_4.
\end{aligned}$$

Similarly, on $\bar{\mathcal{G}}_2$, there exist functions Q_{ij} and W_{3j} such that

$$\begin{aligned} \psi_2 &= Q_{20}\beta_0 + W_{21}\beta_1 + \cdots + Q_{24}\beta_4, \\ \psi_6 &= Q_{60}\beta_0 + W_{61}\beta_1 + \cdots + Q_{64}\beta_4, \\ \mathrm{d}(W_3) &= (\psi_1 + \psi_5)W_3 + W_{30}\beta_0 + W_{31}\beta_1 + \cdots + W_{34}\beta_4. \end{aligned}$$

There is freedom to add linear combinations of $\alpha_0, \ldots, \alpha_4$ (resp. $\beta_0, \ldots, \beta_4$) into φ_i (resp. ψ_i) without changing the form of the corresponding Monge-Ampère structure equations. Using this, we can arrange the following expressions to be zero:

$$P_{21},\ P_{22},\ P_{63},\ P_{64};\ Q_{21},\ Q_{22},\ Q_{63},\ Q_{64}.$$

Applying $\mathrm{d}^2 = 0$ to the Monge-Ampère structure equations yields that

$$\begin{aligned} \mathrm{d}(\mathrm{d}\alpha_1) &\equiv P_{24}V_3\alpha_0 \wedge \alpha_3 \wedge \alpha_4 && \mod \alpha_1, \alpha_2, \\ \mathrm{d}(\mathrm{d}\alpha_2) &\equiv V_{34}\alpha_0 \wedge \alpha_3 \wedge \alpha_4 && \mod \alpha_1, \alpha_2, \\ \mathrm{d}(\mathrm{d}\alpha_3) &\equiv P_{62}V_3\alpha_0 \wedge \alpha_1 \wedge \alpha_2 && \mod \alpha_3, \alpha_4, \\ \mathrm{d}(\mathrm{d}\alpha_4) &\equiv V_{32}\alpha_0 \wedge \alpha_1 \wedge \alpha_2 && \mod \alpha_3, \alpha_4. \end{aligned}$$

This implies that, on $\mathcal{G}_2$,

$$P_{24}V_3 = V_{34} = P_{62}V_3 = V_{32} = 0. \tag{21}$$

By a similar argument, one can show that, on $\bar{\mathcal{G}}_2$,

$$Q_{24}W_3 = W_{34} = Q_{62}W_3 = W_{32} = 0.$$

Restricted to $\mathcal{P}_{\mathrm{IIa}}$, the generators θ_i $(i = 1, \ldots, 4)$ of $\mathcal{J}_1$ satisfy congruences of the form

$$\mathrm{d}\theta_i \equiv -\pi_{i\alpha} \wedge \eta_\alpha + \tau_i \mod \theta_1, \ldots, \theta_4.$$

The tableau takes the form:

$$(\pi_{i\alpha}) = \begin{pmatrix} -\pi_1 & 0 & 0 & 0 & 0 & 0 \\ -\pi_2 & \pi_1 - \pi_5 & 0 & 0 & -s_2\pi_5 + \pi_6 & \pi_6 \\ 0 & 0 & -\pi_3 & 0 & 0 & 0 \\ 0 & 0 & \pi_4 & -\pi_3 + \pi_5 & t_4\pi_5 - \pi_7 & \pi_7 \end{pmatrix},$$

where, modulo η_i,

$$\pi_1 \equiv \varphi_1 - \psi_1, \quad \pi_3 \equiv \varphi_5 - \psi_5, \quad \pi_5 \equiv \varphi_0 - \psi_0, \qquad \pi_7 \equiv \mathrm{d}(t_4) - t_4\psi_5,$$
$$\pi_2 \equiv \varphi_3 - \psi_3, \quad \pi_4 \equiv \varphi_7 + \psi_7, \quad \pi_6 \equiv \mathrm{d}(s_2) - s_2\psi_1.$$

Assuming s_2, t_4 to be both nonzero, we compute and find that the torsion can be absorbed only if the following expressions are zero:

$$P_{20},\ P_{23},\ P_{24},\ P_{60},\ P_{61},\ P_{62};\ Q_{20},\ Q_{23},\ Q_{24},\ Q_{60},\ Q_{61},\ Q_{62}.$$

One can verify that, on the subbundle of $\mathcal{G}_2$ defined by the vanishing of $P_{20}, P_{23}, P_{24}, P_{60}, P_{61}$ and P_{62}, the following structure equations hold:

$$\begin{aligned}
\mathrm{d}\alpha_0 &= -\varphi_0 \wedge \alpha_0 + \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4,\\
\mathrm{d}\alpha_1 &= -\varphi_1 \wedge \alpha_1,\\
\mathrm{d}\alpha_2 &= -\varphi_3 \wedge \alpha_1 + (\varphi_1 - \varphi_0) \wedge \alpha_2 + V_3\alpha_0 \wedge \alpha_3,\\
\mathrm{d}\alpha_3 &= -\varphi_5 \wedge \alpha_3,\\
\mathrm{d}\alpha_4 &= -\varphi_7 \wedge \alpha_3 + (\varphi_5 - \varphi_0) \wedge \alpha_4 + V_3\alpha_0 \wedge \alpha_1.
\end{aligned}$$

Clearly, the systems $\langle\alpha_1\rangle$ and $\langle\alpha_3\rangle$ are both integrable. It is a similar case for the structure equations on $\bar{\mathcal{G}}_2$. By Proposition 2.4, the proof is complete. □

5.2. **Type IIb.** In this case, on a 1-refined lifting of ϕ, either V_1 or V_4 vanishes. By Proposition 4.4 (in particular, using h_J), we can arrange $V_1 \neq 0$ and $V_4 = 0$ on a 1-refined lifting of ϕ. Such a 1-refined lifting can always be chosen to further satisfy $V_1 = 1$ and $V_3 = W_3 = 0$ or 1.

In the next proposition we show that the case of $V_1 = 1$ and $V_3 = W_3 = 0$ is impossible. Then we characterize the case when ϕ admits a 1-refined lifting for which $V_1 = V_3 = W_3 = 1$.

Proposition 5.3. *Restricting to the locus in $\mathcal{P}_1$ defined by*

$$\mu = 1,\ \epsilon = -1,\ V_1 = -W_1 = 1,\ V_2 = V_3 = V_4 = W_2 = W_3 = W_4 = 0,$$

$\mathcal{J}_1$ has no integral manifold.

Proof. If there exists a 1-refined lifting of a special Bäcklund transformation such that $V_3 = W_3 = 0$, then the equality (20) enforces that $s_2 t_4 = 0$ on such a lifting. By Proposition 4.2, both Monge-Ampère systems must be contact equivalent to the wave equation $z_{xy} = 0$. In particular, V_i and W_i must all be zero on $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, respectively. This contradicts our assumption. □

Theorem 5.2. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Euler-Lagrange systems. If $\phi : N \to M \times \bar{M}$ defines a Type IIb special rank-1 Bäcklund transformation relating $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$, then each of $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ must have a characteristic system that contains a rank-1 integrable subsystem.*

Proof. The idea is similar to that of Theorem 5.1. We restrict the differential ideal $\mathcal{J}_1$ to the locus $\mathcal{P}_{\mathrm{IIb}} \subset \mathcal{P}_1$ defined by the equations

$$\mu = 1,\ \epsilon = -1,\ V_1 = -W_1 = V_3 = W_3 = 1,\ V_2 = V_4 = W_2 = W_4 = 0$$

and analyze the obstructions to integrability of the resulting rank-4 Pfaffian system.

The calculations below are performed using Maple™.

By (12), on the subbundle of $\mathcal{G}_1$ defined by $V_1 = V_3 = 1$ and $V_2 = V_4 = 0$, there exist functions P_{ij} such that

$$\begin{aligned}\varphi_2 &= P_{20}\alpha_0 + P_{21}\alpha_1 + \cdots + P_{24}\alpha_4 + \varphi_0 - \varphi_1 + \varphi_5,\\ \varphi_3 &= P_{30}\alpha_0 + P_{31}\alpha_1 + \cdots + P_{34}\alpha_4 + \varphi_1 + \varphi_5,\\ \varphi_6 &= P_{60}\alpha_0 + P_{61}\alpha_1 + \cdots + P_{64}\alpha_4.\end{aligned}$$

Using the freedom in the choice of φ_i, we can arrange that

$$P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{34}, P_{64}$$

are zero.

Expanding $\mathrm{d}(\mathrm{d}\alpha_i) = 0$, we find that

$$P_{24} = 0, \quad P_{61} = -P_{62}, \quad P_{63} = 0.$$

Similarly, on the subbundle of $\bar{\mathcal{G}}_1$ defined by $-W_1 = W_3 = 1$ and $W_2 = W_4 = 0$, there exist functions Q_{ij} such that

$$\begin{aligned}\psi_2 &= Q_{20}\beta_0 + Q_{21}\beta_1 + \cdots + Q_{24}\beta_4 - \psi_0 + \psi_1 - \psi_5,\\ \psi_3 &= Q_{30}\beta_0 + Q_{31}\beta_1 + \cdots + Q_{34}\beta_4 - \psi_1 - \psi_5,\\ \psi_6 &= Q_{60}\beta_0 + Q_{61}\beta_1 + \cdots + Q_{64}\beta_4.\end{aligned}$$

Using the freedom in the choice of ψ_i, we can arrange that

$$Q_{21}, Q_{22}, Q_{23}, Q_{31}, Q_{32}, Q_{34}, Q_{64}$$

are zero.

By expanding $\mathrm{d}(\mathrm{d}\beta_i) = 0$, we find that

$$Q_{24} = 0, \quad Q_{61} = Q_{62}, \quad Q_{63} = 0.$$

Denote the restriction of $\mathcal{J}_1$ to $\mathcal{P}_{\mathrm{IIb}}$ by $\mathcal{J}_{\mathrm{IIb}}$. By computation, we find that the torsion of $(\mathcal{P}_{\mathrm{IIb}}, \mathcal{J}_{\mathrm{IIb}})$ can be absorbed only if the following expressions are zero:

$$s_2t_4 + 1, \quad P_{20} - Q_{20}, \quad P_{60}, \quad P_{62}, \quad Q_{60}, \quad Q_{62}.$$

In particular, the vanishing of P_{60}, P_{62}, Q_{60} and Q_{62} implies that

$$\mathrm{d}\alpha_3 = -\varphi_5 \wedge \alpha_3, \qquad \mathrm{d}\beta_3 = -\psi_5 \wedge \beta_3.$$

The conclusion of the proposition follows. □

6. New Examples of Type III

In this section, we present some new examples of Type III special rank-1 Bäcklund transformations. Their existence shows that a degenerate hyperbolic Euler-Lagrange system may be special Bäcklund-related to a non-degenerate one. One of these examples (Section 6.2) is, up to contact transformations, a Bäcklund transformation relating solutions of the PDE

$$z_{xy} = \frac{2z}{(x+y)^2} \tag{22}$$

to those of a more complicated PDE of the form (3) whose coefficients are given by (42) in Appendix B. We note that (22) is on the Goursat-Vessiot list[10] of PDEs of the form

$$z_{xy} = F(x, y, z, z_x, z_y)$$

that are Darboux-integrable at the 2-jet level.

6.1. **A Class of New Examples.** Using a method in [Hu19] and after a somewhat lengthy calculation using Maple™, we obtain on a 6-manifold N involutive[11] structure equations (see Appendix A for details) of the form:

$$\mathrm{d}\omega^i = -\frac{1}{2}C^i_{jk}(R, S, T)\omega^j \wedge \omega^k, \tag{23}$$

$$\begin{aligned} \mathrm{d}R &= R_k(R, S, T)\omega^k, \\ \mathrm{d}S &= S_k(R, S, T)\omega^k, \\ \mathrm{d}T &= T_k(R, S, T)\omega^k, \end{aligned} \tag{24}$$

where $C^i_{jk} = -C^i_{kj}, R_k, S_k, T_k$ are certain fixed analytic functions defined on $\Omega := \{RST \neq 0\} \subset \mathbb{R}^3$.

One can verify that each of the ideals

$$\mathcal{I} := \langle \omega^1, \omega^3 \wedge \omega^4, \omega^5 \wedge \omega^6 \rangle_{\mathrm{alg}}, \quad \bar{\mathcal{I}} := \langle \omega^2, \omega^3 \wedge \omega^4, \omega^5 \wedge \omega^6 \rangle_{\mathrm{alg}}$$

is generated by the pull-back of a hyperbolic Monge-Ampère ideal ($\mathcal{I}_0$ and $\bar{\mathcal{I}}_0$, resp.) defined on a 5-dimensional quotient (M and $\bar{M}$, resp.) of N.

By

$$\begin{aligned} \mathrm{d}\omega^1 &\equiv -R\omega^3 \wedge \omega^4 + \omega^5 \wedge \omega^6, \quad \mod \omega^1, \omega^2, \\ \mathrm{d}\omega^2 &\equiv \omega^3 \wedge \omega^4 + \frac{1}{R}\omega^5 \wedge \omega^6, \quad \mod \omega^1, \omega^2, \end{aligned}$$

it is immediate that N is a special rank-1 Bäcklund transformation relating $(M, \mathcal{I}_0)$ and $(\bar{M}, \bar{\mathcal{I}}_0)$. Furthermore, we choose the new bases

$$(\sigma^0, \sigma^1, \ldots, \sigma^4) = (\omega^1, -R\omega^3, \omega^4 - \omega^1, \omega^5, \omega^6 + R^{-1}\omega^1) \tag{25}$$

and

$$(\tau^0, \tau^1, \ldots, \tau^4) = (\omega^2, \omega^3, \omega^4 - R\omega^2, R^{-1}\omega^5, \omega^6 - \omega^2), \tag{26}$$

such that their pull-backs via arbitrary sections $\iota : M \to N$ and $\bar{\iota} : \bar{M} \to N$, respectively, are 1-adapted hyperbolic Monge-Ampère coframings.

Computing using these new bases, we find that

$$S_1 = \begin{pmatrix} 2S & -R^{-1}T \\ 4T^{-1}S^2R & -2S \end{pmatrix}, \quad S_2 = \mathbf{0},$$

$$\bar{S}_1 = \begin{pmatrix} -2RS & R^{-1}T \\ 2T^{-1}R(2S^2R^2 - T) & -2RS \end{pmatrix}, \quad \bar{S}_2 = \mathbf{0}.$$

[10] See [Gou99], [Ves39] and [Ves42]. The recent [CI09] has a summary of the list.

[11] Namely, exterior differentiation applied to these equations yields identities.

Clearly, this tells us that both $\mathcal{I}_0$ and $\bar{\mathcal{I}}_0$ are hyperbolic Euler-Lagrange (Proposition 2.2). Moreover, since $\det(S_1) = 0$ and $\det(\bar{S}_1) = 2T \neq 0$, $\mathcal{I}_0$ is degenerate, and $\bar{\mathcal{I}}_0$ is non-degenerate, by our definition.

By Cartan's third theorem (see [Bry14]), for each $u \in \mathbb{R}^6$ and for any $(R_0, S_0, T_0) \in \Omega$, there exists a coframing (ω^i) and functions R, S, T on a neighborhood U of u such that the structure equations (23) and (24) hold with $(R, S, T)(u) = (R_0, S_0, T_0)$.

This fact tells us that T may be positive or negative. In other words, $\bar{\mathcal{I}}_0$ can be a hyperbolic Euler-Lagrange system of either the positive or the negative type.

Note that, generically, the map $(R, S, T) : N^6 \to \mathbb{R}^3$ has rank 3. This is easy to see by computing the differentials $\mathrm{d}(R), \mathrm{d}(S)$ and $\mathrm{d}(T)$. In this generic case, N represents a Bäcklund transformation of cohomogeneity 3.

On the other hand, the map (R, S, T) has rank 2, which is the minimum rank possible, if and only if

$$T = 2R^2S^2.$$

By the structure equations, $\mathrm{d}(T - 2R^2S^2)$ vanishes whenever $T - 2R^2S^2$ does.

This gives rise to the condition when the underlying Bäcklund transformation has cohomogeneity 2, that is, when $T = 2R^2S^2$. In particular, since $T > 0$ in this case, $\bar{\mathcal{I}}_0$ must be of the positive type. We present further analysis of this case below.

6.2. The $T = 2R^2S^2$ Case.

All calculations below, unless otherwise noted, are performed using Maple™.

Setting $T = 2R^2S^2$, the structure equations (23) and (24) now take the form

$$\mathrm{d}\omega^i = -\frac{1}{2}\tilde{C}^i_{jk}(R, S)\omega^j \wedge \omega^k, \tag{27}$$

$$\begin{aligned} \mathrm{d}R &= \tilde{R}_k(R, S)\omega^k, \\ \mathrm{d}S &= \tilde{S}_k(R, S)\omega^k, \end{aligned} \tag{28}$$

where $\tilde{C}^i_{jk}(R, S) = C^i_{jk}(R, S, 2R^2S^2)$, and similarly for $\tilde{R}_k$ and $\tilde{S}_k$.

Let σ^i and τ^i be as in (25) and (26), respectively. We find that the 1-form

$$\phi := -\frac{\omega^2}{2S} - \frac{1}{2SR}\sigma^0 - \frac{1}{2}\sigma^1 + SR\sigma^2 - \frac{R}{2}\sigma^3 + \frac{R^2S^2 - S^2 + 1}{S}\sigma^4$$

is exact. Thus, locally there exists a function f on N such that

$$\mathrm{d}f = \phi.$$

Now consider the 1-forms

$$(\xi^0, \xi^1, \ldots, \xi^4) := \left(\frac{\sigma^0}{e^f R},\ \frac{\sigma^1 - RS\sigma^2}{R},\ \frac{\sigma^2}{e^f},\ \sigma^3 - RS\sigma^4,\ \frac{\sigma^4}{e^f R}\right).$$

Using (27) and (28) and by letting $U = e^f/S$, we obtain the following structure equations for (ξ^i):

$$\mathrm{d}\xi^0 = \xi^1 \wedge \xi^2 + \xi^3 \wedge \xi^4, \tag{29}$$

$$\mathrm{d}\xi^1 = -U\xi^0 \wedge \xi^1 + \frac{U}{2R}\xi^1 \wedge \xi^2 + \frac{R^2-1}{2R}\xi^1 \wedge \xi^3 - \frac{UR}{2}\xi^1 \wedge \xi^4, \tag{30}$$

$$\begin{aligned}\mathrm{d}\xi^2 &= 2\xi^0 \wedge \xi^1 + U\xi^0 \wedge \xi^2 + 2\xi^0 \wedge \xi^3 - \frac{R^2+1}{2R}\xi^1 \wedge \xi^2 \\ &\quad + R\xi^1 \wedge \xi^4 - \frac{R^2-1}{2R}\xi^2 \wedge \xi^3 + \frac{UR}{2}\xi^2 \wedge \xi^4,\end{aligned} \tag{31}$$

$$\mathrm{d}\xi^3 = U\xi^0 \wedge \xi^3 - \frac{R^2-1}{2R}\xi^1 \wedge \xi^3 + \frac{U}{2R}\xi^2 \wedge \xi^3 + \frac{UR}{2}\xi^3 \wedge \xi^4, \tag{32}$$

$$\begin{aligned}\mathrm{d}\xi^4 &= 2\xi^0 \wedge \xi^1 + 2\xi^0 \wedge \xi^3 - U\xi^0 \wedge \xi^4 + \frac{R^2-1}{2R}\xi^1 \wedge \xi^4 \\ &\quad + \frac{1}{R}\xi^2 \wedge \xi^3 - \frac{U}{2R}\xi^2 \wedge \xi^4 + \frac{R^2+1}{2R}\xi^3 \wedge \xi^4,\end{aligned} \tag{33}$$

$$\mathrm{d}(U) = -RU\xi^1 + \frac{U}{R}\xi^3, \tag{34}$$

$$\mathrm{d}(R) = UR\xi^0 + \frac{1}{2}(R^2+1)(\xi^1 + \xi^3) + \frac{U}{2}\xi^2 - \frac{R^2U}{2}\xi^4. \tag{35}$$

It is clear from these structure equations that ξ^i $(i = 0, \ldots, 4)$ are well-defined on a 5-dimensional quotient M^5 of N, the corresponding hyperbolic Euler-Lagrange system being $(M, \mathcal{I}_0)$, where

$$\mathcal{I}_0 = \langle \xi^0, \xi^1 \wedge \xi^2, \xi^3 \wedge \xi^4 \rangle_{\mathrm{alg}}.$$

In particular, note that ξ^1 and ξ^3 are both integrable. By Proposition 2.4, $(M, \mathcal{I}_0)$ is contact equivalent to a PDE of the form $z_{xy} = F(x, y, z, z_x, z_y)$.

In Appendix B, we find local coordinates for $(M, \mathcal{I}_0)$ and prove that it is contact equivalent to the equation:

$$z_{xy} = \frac{2z}{(x+y)^2}.$$

On the other hand, consider the 1-forms

$$(\eta^0, \eta^1, \ldots, \eta^4) := \\ \left(RS\tau^0,\ R\tau^1 + \frac{R(R^2S^2-1)}{S(R^2+1)}\tau^2,\ S\tau^2,\ R\tau^3 - \frac{R(R^2S^2+1)}{S(R^2+1)}\tau^4,\ S\tau^4\right).$$

Using (27) and (28), we find that

$$
\begin{aligned}
(36)\qquad \mathrm{d}\eta^0 &= \frac{R^2+1}{2R^2}\eta^0\wedge\left(\eta^1-\frac{R(2R^2-1)}{R^2+1}\eta^2+R\eta^3+\frac{R^2}{R^2+1}\eta^4\right)\\
&\quad+\eta^1\wedge\eta^2+\eta^3\wedge\eta^4,\\
(37)\qquad \mathrm{d}\eta^1 &= -2\eta^0\wedge\left(\frac{\eta^1}{R}+\eta^2+\eta^3+\frac{R(R^2-1)}{R^2+1}\eta^4\right)-\frac{R(2R^2+1)}{R^2+1}\eta^1\wedge\eta^2\\
&\quad-\frac{1}{R}\eta^1\wedge\eta^3+\frac{2}{R^2+1}\eta^1\wedge\eta^4-\frac{R^2+2}{R^2+1}\eta^2\wedge\eta^3+\frac{2R}{(R^2+1)^2}\eta^2\wedge\eta^4,\\
(38)\qquad \mathrm{d}\eta^2 &= \eta^0\wedge\left(\frac{R^2+1}{R}\eta^2-2\eta^4\right)-\frac{3R^2-1}{2R^2}\eta^1\wedge\eta^2-\frac{1}{R}\eta^1\wedge\eta^4\\
&\quad+\frac{R^2+3}{2R}\eta^2\wedge\eta^3+\frac{R^2-3}{2(R^2+1)}\eta^2\wedge\eta^4,\\
(39)\qquad \mathrm{d}\eta^3 &= \eta^0\wedge\left(2\eta^1-\frac{2R(R^2-1)}{R^2+1}\eta^2+\frac{R^2-1}{R}\eta^3\right)+\frac{R^2-1}{2R^2}\eta^1\wedge\eta^3\\
&\quad-\frac{R}{R^2+1}\eta^1\wedge\eta^4+\frac{3R^2-1}{2R(R^2+1)}\eta^2\wedge\eta^3+\frac{R^2(R^2-1)}{(R^2+1)^2}\eta^2\wedge\eta^4\\
&\quad+\frac{3R^2+1}{2(R^2+1)}\eta^3\wedge\eta^4,\\
(40)\qquad \mathrm{d}\eta^4 &= 2\eta^0\wedge\eta^2-\eta^1\wedge\eta^4+\eta^2\wedge\eta^3+\frac{R(R^2-1)}{(R^2+1)}\eta^2\wedge\eta^4-\frac{2}{R}\eta^3\wedge\eta^4,\\
(41)\qquad \mathrm{d}(R) &= -(R^2+1)\eta^0-\frac{R^2+1}{2R}\eta^1-\frac{1}{2}\eta^2+\frac{R^2+1}{2}\eta^3-\frac{R}{2}\eta^4.
\end{aligned}
$$

It is clear, by these structure equations, that (ξ^i) descend to a coframing on a 5-dimensional quotient $\bar{M}^5$ of N, the corresponding hyperbolic Euler-Lagrange system being $(\bar{M}^5,\bar{\mathcal{I}}_0)$, where

$$\bar{\mathcal{I}}_0=\langle\eta^0,\eta^1\wedge\eta^2,\eta^3\wedge\eta^4\rangle_{\mathrm{alg}}.$$

We have found local coordinates on $\bar{M}$ that put $(\bar{M},\bar{\mathcal{I}}_0)$ (up to contact transformations) in a PDE form. Since both the expression and its derivation are rather complicated, we include them in Appendix B.

Remark 5. This example shows the possibility for a pair of hyperbolic Euler-Lagrange systems of distinct types to be special Bäcklund-related, in which one admits nontrivial first integrals for its characteristic systems, while the other doesn't. In contrast, suppose that S and $\bar{S}$ are, respectively, integral surfaces of two hyperbolic Monge-Ampère systems $(M,\mathcal{I})$ and $(\bar{M},\bar{\mathcal{I}})$. If $S,\bar{S}$ are related to each other by a rank-1 Bäcklund transformation, then the characteristic curves in S correspond, under the Bäcklund transformation, to the characteristic curves in $\bar{S}$, and *vice versa*. This is easy to see by the condition (2) in Definition 1.3.

7. Some Classical Examples

In this section, we present some examples of special rank-1 Bäcklund transformations. These examples are not new. We hope they can serve as a motivation for the open questions described in the next section.

1. *A class of homogeneous rank-*1 *Bäcklund transformations.* In [Cle02], it is shown that, if, on N^6, there exists a local coframing $(\theta_1, \theta_2, \omega^1, \dots, \omega^4)$ satisfying

$$\begin{aligned}
\mathrm{d}\theta_1 &= \theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4,\\
\mathrm{d}\theta_2 &= -\theta_1 \wedge (\omega^1 + \omega^3) + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4,\\
\mathrm{d}\omega^1 &= (B_1(\theta_1 + \theta_2) - \sigma\omega^4) \wedge \omega^2 + B_1\theta_1 \wedge \theta_2,\\
\mathrm{d}\omega^2 &= (-\sigma B_3(\theta_1 + \theta_2) + B_1^{-1}B_3\omega^4) \wedge \omega^1 + \omega^3 \wedge \omega^4,\\
\mathrm{d}\omega^3 &= (-B_3(\theta_1 - \theta_2) - \sigma\omega^2) \wedge \omega^4 + B_3\theta_1 \wedge \theta_2,\\
\mathrm{d}\omega^4 &= (\sigma B_1(\theta_1 - \theta_2) + B_3^{-1}B_1\omega^2) \wedge \omega^3 + \omega^1 \wedge \omega^2,
\end{aligned}$$

where B_i are constants and $\sigma = \pm 1$, then the systems

$$\langle \theta_1, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle_{\mathrm{alg}}, \quad \langle \theta_2, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle_{\mathrm{alg}}$$

are hyperbolic Monge-Ampère systems well-defined on some 5-dimensional quotients of N. Here N is a homogeneous rank-1 Bäcklund transformation relating the two Monge-Ampère systems.

It is clear, by the equation (see Lemma 3.1)

$$(\mathrm{d}\theta_1 + \lambda \mathrm{d}\theta_2)^2 \equiv 0 \mod \theta_1, \theta_2,$$

that $\epsilon\mu^4 = -1$. Therefore, these Bäcklund transformations are special.

In particular, when $\sigma = -1$, $B_1 = -(\tau + 1)^2/2$ and $B_3 = (\tau^{-1} + 1)^2/2$ for some constant $\tau \neq 0$, one obtains the classical Bäcklund transformation between $K = -1$ surfaces in $\mathbb{E}^3$ when $\tau > 0$ and a Bäcklund transformation between $K = 1$ surfaces in $\mathbb{E}^{2,1}$ when $\tau < 0$.

2. Let (x, y, u, p, q) and (X, Y, v, P, Q) be coordinates on the spaces of two sine-Gordon systems:

$$\begin{aligned}
\mathcal{I} &= \left\langle \mathrm{d}u - p\mathrm{d}x - q\mathrm{d}y, \left(\mathrm{d}p - \frac{1}{2}\sin(2u)\mathrm{d}y\right) \wedge \mathrm{d}x \right\rangle,\\
\bar{\mathcal{I}} &= \left\langle \mathrm{d}v - P\mathrm{d}X - Q\mathrm{d}Y, \left(\mathrm{d}P - \frac{1}{2}\sin(2v)\mathrm{d}Y\right) \wedge \mathrm{d}X \right\rangle.
\end{aligned}$$

The locus N in the product manifold defined by the equations

$$X = x, \quad Y = y, \quad p = P + \lambda\sin(u + v), \quad Q = -q + \lambda^{-1}\sin(u - v)$$

corresponds to the Bäcklund transformation (2).

The pull-backs to N of $\theta = \mathrm{d}u - p\mathrm{d}x - q\mathrm{d}y$ and $\bar{\theta} = \mathrm{d}v - P\mathrm{d}X - Q\mathrm{d}Y$ satisfy

$$(\mathrm{d}\theta + \mathrm{d}\bar{\theta})^2 \equiv (\mathrm{d}\theta - \mathrm{d}\bar{\theta})^2 \equiv 0 \mod \theta, \bar{\theta}.$$

It follows that the Bäcklund transformation (2) is special.

8. Open Questions

Several results concerning special Bäcklund transformations relating two hyperbolic Euler-Lagrange systems can be summarized in the diagram (Figure 1) below.

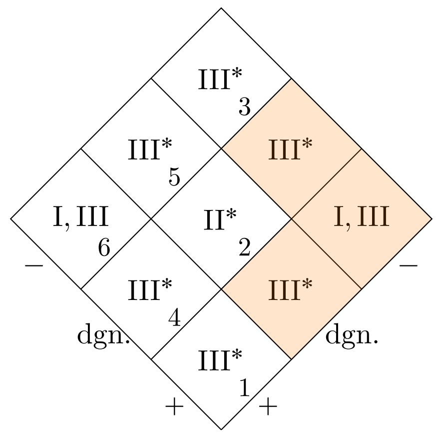


Figure 1. These are possible types of special rank-1 Bäcklund transformations relating two hyperbolic Euler-Lagrange systems. The types of an Euler-Lagrange system (positive, degenerate, negative) are denoted by +, dgn., −, respectively. The superscript * indicates that example(s) exist and are either mentioned or discovered in this paper. By symmetry, one only needs to consider regions 1 to 6 in the diagram.

- **a.** An immediate question is: *Can a special rank-*1 *Bäcklund transformation belong to region* 6 *in the diagram?*
- **b.** Does there exist any special rank-1 Bäcklund transformation in regions 1 and 3 that does not relate an Euler-Lagrange system to itself?
- **c.** A more subtle problem is to understand the generality of existence in each of the regions. For instance, it would be interesting to determine integers k, ℓ such that: *The space of special rank-*1 *Bäcklund transformations relating two positive hyperbolic Euler-Lagrange systems is parametrized by* k *functions of* ℓ *variables.*
- **d.** What can be said about the existence of non-special rank-1 Bäcklund transformations relating two hyperbolic Euler-Lagrange systems? Is there any hyperbolic Euler-Lagrange pair that are rank-1 Bäcklund-related but not special rank-1 Bäcklund-related?
- **e.** When can a hyperbolic Euler-Lagrange system be Bäcklund-related to a non-Euler-Lagrange hyperbolic Monge-Ampère system?

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Appendix A. Table of Structure Functions

In this appendix, we record the functions $C^i_{jk} = -C^i_{kj}, R_k, S_k, T_k$ involved in the equations (23) and (24).

Table 1: Expressions of C^i_{jk}, R_k, S_k, T_k in (23) and (24). All C^i_{jk} $(j < k)$ and R_k, S_k, T_k that are not on this list are zero.

C^1_{12}	$-\frac{SR^2}{T}$	C^1_{13}	$\frac{2R^2-1}{2R}$
C^1_{14}	$\frac{R^2T+2R^2-T}{4SR^3}$	C^1_{15}	$\frac{4R^2S^2+T}{2RT}$
C^1_{16}	$-S$	C^1_{34}	R
C^1_{56}	-1	C^2_{12}	$\frac{4R^4S^2T+4R^4S^2-R^2T^2-2R^2T+T^2}{4SR^3T}$
C^2_{23}	$\frac{R(4R^2S^2-3T)}{2T}$	C^2_{24}	SR
C^2_{25}	$-\frac{R^2+2}{2R}$	C^2_{26}	$\frac{R^2T+2R^2-T}{4SR^2}$
C^2_{34}	-1	C^2_{56}	$-\frac{1}{R}$
C^3_{12}	-1	C^3_{13}	$\frac{2R^4S^2T+4R^4S^2-2R^2S^2T-R^2T^2-2R^2T+T^2}{2SR^3T}$
C^3_{14}	$\frac{1}{R}$	C^3_{24}	-1
C^3_{35}	$\frac{4R^2S^2-R^2T-T}{2RT}$	C^3_{34}	$\frac{4R^4S^2-R^2T-2R^2+T}{4SR^3}$
C^3_{45}	$\frac{2S}{R}$	C^3_{46}	$-\frac{T}{R^2}$
C^3_{36}	$-\frac{4R^2S^2-R^2T-2R^2+T}{4SR^2}$	C^4_{13}	$\frac{2(2R^2S^2-T)}{RT}$
C^4_{24}	$\frac{SR^2}{T}$	C^4_{14}	$-\frac{4R^4S^2-4R^2S^2T-R^2T^2-2R^2T+T^2}{4SR^3T}$
C^4_{34}	$-\frac{1}{2R}$	C^4_{36}	1
C^4_{45}	$-\frac{4R^2S^2-T}{2RT}$	C^4_{46}	S
C^4_{56}	-1	C^5_{15}	$-\frac{2R^4S^2T+4R^4S^2-2R^2S^2T-R^2T^2-2R^2T+T^2}{2SR^3T}$
C^5_{12}	-1	C^5_{16}	1
C^5_{26}	R	C^5_{36}	$2SR^2$

C^5_{46}	T	C^5_{45}	$-\frac{4R^4S^2+R^2T+2R^2-T}{4SR^3}$
C^5_{35}	$-\frac{4R^4S^2-R^2T-T}{2RT}$	C^5_{56}	$-\frac{4R^2S^2+R^2T+2R^2-T}{4SR^2}$
C^6_{15}	$-\frac{2(2R^2S^2-T)}{T}$	C^6_{16}	$\frac{4R^4S^2T+4R^4S^2-R^2T^2-2R^2T+T^2}{4SR^3T}$
C^6_{26}	$\frac{SR^2}{T}$	C^6_{34}	-1
C^6_{36}	$\frac{R(4R^2S^2-T)}{2T}$	C^6_{45}	-1
C^6_{46}	SR	C^6_{56}	$-\frac{R}{2}$
R_3	$-\frac{4R^4S^2-R^2T+T}{2T}$	R_1	$\frac{2R^4S^2T+4R^4S^2-2R^2S^2T-R^2T^2-2R^2T+T^2}{2SR^2T}$
R_4	$-\frac{4R^4S^2-R^2T-2R^2+T}{4SR^2}$	R_5	$\frac{4R^2S^2+R^2T-T}{2T}$
R_6	$-\frac{4R^2S^2+R^2T+2R^2-T}{4SR}$	S_1	$\frac{4R^4S^2T-4R^4S^2+8R^2S^2T-R^2T^2+2R^2T-3T^2}{4R^3T}$
S_2	$-\frac{S^2R^2}{T}$	S_3	$-\frac{SR(4R^2S^2-T)}{2T}$
S_4	$-\frac{2R^2S^2-T}{2R}$	S_5	$-\frac{S(8R^2S^2+R^2T-2T)}{2RT}$
S_6	$\frac{8R^2S^2+R^2T+2R^2-3T}{4R^2}$	T_1	$\frac{4R^4S^2T+4R^4S^2-R^2T^2-2R^2T+T^2}{2SR^3}$
T_2	$-2SR^2$	T_3	$-\frac{4R^4S^2+T}{R}$
T_4	$-\frac{T(4R^4S^2-R^2T-2R^2+T)}{2SR^3}$	T_5	$-\frac{T}{R}$

(End of Table 1)

Appendix B. $(M, \mathcal{I}_0)$ and $(\bar{M}, \bar{\mathcal{I}}_0)$ in PDE Forms

In this Appendix, we integrate the structure equations (29)-(35) for $(M, \mathcal{I}_0)$ and (36)-(41) for $(\bar{M}, \bar{\mathcal{I}}_0)$. By 'integrate', we mean finding local coordinates, expressing the coframings and structure functions in terms of these coordinates such that the structure equations hold identically. As a result, we put these hyperbolic Euler-Lagrange systems in PDE forms (up to contact transformations). Most calculations below are performed using Maple$^{\text{TM}}$.

I. Integration of $(M, \mathcal{I}_0)$.

Consider (29)-(35). We integrate these structure equations in the following steps.

1) First note that these structure equations are invariant under the transformation:

$$(\xi^0, \xi^1, \xi^3) \mapsto (-\xi^0, -\xi^1, -\xi^3), \qquad (U, R) \mapsto (-U, -R).$$

As a result, we can assume that $R > 0$.

2) It is clear that

$$\mathrm{d}\xi^i \equiv 0 \mod \xi^i \qquad (i = 1, 3).$$

Thus, both ξ^1 and ξ^3 are multiples of exact forms. In fact, we find that the 1-forms

$$\psi_1 = -U\xi^0 - \frac{3R^2+1}{2R}\xi^1 - \frac{U}{2R}\xi^2 - \frac{R^2-1}{2R}\xi^3 + \frac{UR}{2}\xi^4,$$
$$\psi_3 = U\xi^0 - \frac{R^2-1}{2R}\xi^1 + \frac{U}{2R}\xi^2 + \frac{R^2+3}{2R}\xi^3 - \frac{UR}{2}\xi^4$$

satisfy

$$\mathrm{d}\xi^1 = \psi_1 \wedge \xi^1, \quad \mathrm{d}\xi^3 = \psi_3 \wedge \xi^3$$

and

$$\mathrm{d}\psi_1 = \mathrm{d}\psi_3 = 0.$$

It follows that there exist local coordinates x, y, r, s such that

$$\xi^1 = e^r \mathrm{d}x, \quad \xi^3 = e^s \mathrm{d}y; \quad \mathrm{d}r = \psi_1, \quad \mathrm{d}s = \psi_3,$$

with freedom to add constants to r, s, x, y, simultaneously scaling x, y.

3) It is straightforward to verify that

$$\mathrm{d}(e^{r+s}/U^2) = \mathrm{d}(e^r R/U) = 0.$$

Using the freedom of adding a constant to r or s, we can arrange that

$$e^{r+s} = U^2;$$

moreover, there exists a constant $\lambda \neq 0$ such that $e^r R = \lambda U$. (At this point, we choose not to normalize λ.) Using these and the expression of $\mathrm{d}U$, we obtain

$$\mathrm{d}U = -U^2\lambda \mathrm{d}x + \frac{U^2}{\lambda}\mathrm{d}y.$$

Direct integration gives

$$U = \frac{1}{\lambda x - \lambda^{-1} y + C}$$

for some constant C. We can normalize C to be zero by adding a constant to x or y.

4) Since

$$\mathrm{d}\xi^i \equiv 0 \mod \xi^0, \xi^2, \xi^4 \qquad (i = 0, 2, 4),$$

the system $\langle \xi^0, \xi^2, \xi^4 \rangle$ is Frobenius. We proceed by looking for its first integrals.

Indeed, we find that

$$\Phi := \xi^0 + \frac{1}{2R}\xi^2 - \frac{R}{2}\xi^4$$

satisfies

$$\mathrm{d}\Phi = \frac{R^2-1}{R}(\xi^1 + \xi^3) \wedge \Phi.$$

Hence, Φ is a multiple of an exact form. By adding an appropriate multiple of Φ into $\frac{R^2-1}{R}(\xi^1+\xi^3)$, we find that

$$\mathrm{d}(-2\ln R + 2\ln(R^2+1)) \equiv \frac{R^2-1}{R}(\xi^1+\xi^3) \mod \Phi.$$

Therefore, locally there exists a function v such that

$$\Phi = \left(\frac{R^2+1}{R}\right)^2 \mathrm{d}v.$$

5) One can verify that the functions R, x, y, v do not have linearly independent differentials. In fact, we find that

$$\mathrm{d}\left(\frac{1}{R^2+1}\right) = -\frac{\lambda U}{R^2+1}\mathrm{d}x - \frac{R^2 U}{R^2+1}\mathrm{d}y - 2U\mathrm{d}v.$$

Setting $G(x,y,z) := (R^2+1)^{-1}$, direct integration gives

$$G = \frac{1}{R^2+1} = -2Uv + \frac{-y+C_1}{\lambda^2 x - y},$$

where C_1 is a constant. We can normalize C_1 to be zero by adding constants to x and y. For convenience, we express v in terms of R, x, y.

6) Next, we observe that the 1-forms

$$\Xi_2 := \frac{1}{R}\xi^2 + \frac{\lambda}{R}\mathrm{d}x - \frac{2\ln R}{\lambda}\mathrm{d}y, \qquad \Xi_4 := R\xi^4 - 2\lambda\ln R\ \mathrm{d}x - \frac{R^2}{\lambda}\mathrm{d}y$$

satisfy

$$\mathrm{d}(\Xi_2+\Xi_4) = 0, \qquad \mathrm{d}((\lambda^2 x - y)\Xi_2) = -\mathrm{d}y \wedge (\Xi_2+\Xi_4).$$

Consequently, there exist functions g, h such that

$$\Xi_2 + \Xi_4 = \mathrm{d}g, \qquad \Xi_2 = \frac{\mathrm{d}h - y\mathrm{d}g}{\lambda^2 x - y}.$$

7) At this point, each ξ^i as well as the function U can be expressed in terms of x, y, R, g, h and their exterior derivatives. Moreover, (29)-(35) become identities.

In particular, one can already put ξ^0 in the form

$$\xi^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,$$

by introducing certain functions z, p, q. In our choice,

$$z = \frac{\lambda^2 x - y}{\lambda}\ln R + \frac{(\lambda^2 x + y)g}{2(\lambda^2 x - y)} - \frac{h}{\lambda^2 x - y}.$$

Computing using these coordinates, we obtain

$$\xi^1 \wedge \xi^2 + \mathrm{d}p \wedge \mathrm{d}x = \frac{2\lambda^2 z}{(\lambda^2 x - y)^2}\mathrm{d}x \wedge \mathrm{d}y.$$

This tells us that $(M, \mathcal{I}_0)$ is contact equivalent to the PDE

$$z_{xy} = -\frac{2\lambda^2 z}{(\lambda^2 x - y)^2},$$

which is, after we scale x and y, contact equivalent to

$$z_{xy} = \frac{2z}{(x+y)^2}.$$

II. Integration of $(\bar{M}, \bar{\mathcal{I}}_0)$.

We integrate the structure equations (36)-(41) in the following steps.

1) First note that

$$\mathrm{d}\eta^i \equiv 0 \mod \eta^2, \eta^4 \qquad (i = 2, 4).$$

Thus, the system $\langle \eta^2, \eta^4 \rangle$ is Frobenius. In particular, there exists a linear combination of η^2, η^4 that is a multiple of an exact 1-form. Setting undetermined weights that are functions of R, we find that the 1-form

$$\Phi := \eta^2 - R\eta^4$$

satisfies

$$\mathrm{d}\Phi = \psi \wedge \Phi$$

for some 1-form ψ.

By adding an appropriate multiple of Φ to ψ, we obtain an exact 1-form (still denoted by ψ):

$$\psi = -\frac{R^2-1}{R}\eta^0 - \frac{3R^2-1}{2R^2}\eta^1 + \frac{R^2-3}{2R}\eta^3 + \frac{(R^2-1)^2}{R^2+1}\eta^4 + \frac{2R^4 - R^2 + 1}{2R(R^2+1)}\Phi.$$

Thus, locally there exist functions f, g such that

$$\mathrm{d}f = \psi, \qquad \Phi = e^f \mathrm{d}g.$$

2) Next, we compute and observe that

$$\mathrm{d}\eta^4 \equiv \left(-\frac{2(2R^2-1)}{R(R^2+1)}\mathrm{d}R + 2\mathrm{d}f \right) \wedge \eta^4 \mod \mathrm{d}g.$$

This motivates the calculation:

$$\mathrm{d}\left(\frac{(R^2+1)^3}{e^{2f}R^2}\eta^4 \right) = \gamma \wedge \mathrm{d}g,$$

where

$$\gamma = \frac{(R^2+1)^3}{R^4 e^f}\eta^3 + \frac{(R^2+1)^2}{Re^f}\eta^4 - \frac{(R^2+1)^2(3R^2-1)}{R^4 e^f}\mathrm{d}R + \frac{(R^2+1)^3}{R^3 e^f}\mathrm{d}f + (R^2 + 4\ln|R| - R^{-2})\mathrm{d}g.$$

We constructed this γ, by choosing an appropriate '$\mathrm{d}g$'-term, so that it is exact.

Consequently, there exist functions u, v such that

$$\mathrm{d}u = \gamma, \qquad \mathrm{d}v = \frac{(R^2+1)^3}{e^{2f}R^2}\eta^4 - u\mathrm{d}g.$$

3) Now each η^i can be expressed completely in terms of f, g, R, u, v and their differentials. Moreover, (36)-(41) become identities.

   In particular,

$$\begin{aligned}\eta^0 = &\frac{3R^2-1}{R^2+1}\mathrm{d}R - \frac{R^2e^f((R^2+1)^2+2(R^2+1)\ln|R|+e^f uR)}{(R^2+1)^3}\mathrm{d}g\\ &+\frac{R^2e^f}{2(R^2+1)^2}\mathrm{d}u - \frac{e^{2f}R^3}{(R^2+1)^3}\mathrm{d}v - \frac{R}{2}\mathrm{d}f.\end{aligned}$$

   By the Pfaff theorem, up to scaling, we can put η^0 in the form

$$\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y.$$

   We approach this by following the proof of the Pfaff theorem in Chapter II of [BCG⁺13]. We first expand $\mathrm{d}\eta^0 \wedge \eta^0 \wedge \mathrm{d}R$ in coordinates; from its expression we find that

$$x := 2(R^2 + 2\ln|R| + 1)g - u$$

   satisfies

$$\mathrm{d}\eta^0 \wedge \eta^0 \wedge \mathrm{d}R \wedge \mathrm{d}x = 0.$$

   Next, writing u in terms of x, R and g, the 3-form $\eta^0 \wedge \mathrm{d}R \wedge \mathrm{d}x$ has only 3 terms in it. This allows us to find

$$y := -v - (R^2+1+2\ln|R|)g^2 + xg + \frac{1}{4R^2e^{2f}}(R^2+1)^3,$$

   which satisfies

$$\eta^0 \wedge \mathrm{d}R \wedge \mathrm{d}x \wedge \mathrm{d}y = 0.$$

4) It follows from the previous step that η^0 is a linear combination of $\mathrm{d}x, \mathrm{d}y$ and $\mathrm{d}R$. After scaling η^0 such that the coefficient of $\mathrm{d}R$ is 1, we obtain the 1-form

$$\mathrm{d}R - p\mathrm{d}x - q\mathrm{d}y,$$

   where

$$p = \frac{(2gRe^f + R^2 + 1)Re^f}{(R^2+1)G}, \qquad q = -\frac{2R^2e^{2f}}{(R^2+1)G},$$

   and

$$G = 4ge^f(gRe^f + R^2 + 1) + R^3 + R.$$

5) It is now reasonable to set $z = R$. Using these coordinates, we compute and observe that

$$\begin{aligned}\eta^1 \wedge \eta^2 \equiv\, &A\mathrm{d}p \wedge \mathrm{d}q + B\mathrm{d}p \wedge \mathrm{d}y + C(\mathrm{d}x \wedge \mathrm{d}p - \mathrm{d}y \wedge \mathrm{d}q)\\ &+ D\mathrm{d}x \wedge \mathrm{d}q + E\mathrm{d}x \wedge \mathrm{d}y \qquad \mod \eta^0, \mathrm{d}\eta^0,\end{aligned}$$

where

$$\begin{aligned} A &= 2qz(z^2+1)^3, \\ B &= 2q^2(z^2+1)^2(4p^2z^3 - qz^2 + 4p^2z + 3q), \\ C &= -2pq(z^2+1)^3(4p^2z+q), \\ D &= (z^2+1)(4p^2z^3 + qz^2 + 4p^2z - q)(2p^2z^2 + qz + 2p^2), \\ E &= -q^3(4p^2z^5 + qz^4 - 16p^2z^3 - 8qz^2 - 20p^2z - q). \end{aligned} \tag{42}$$

The PDE form of $(\bar{M}, \bar{\mathcal{I}}_0)$ (up to contact transformations) is therefore (3) in the introduction, where A, B, C, D, E are the same as the above with p, q being replaced by z_x and z_y, respectively.

Furthermore, hyperbolicity demands that

$$0 < AE - BD + C^2 = 8q^4(2p^2z^2 + 2p^2 + zq)(z^2+1)^4;$$

in other words, the domain of the variables (x, y, z, p, q) needs to satisfy $q \neq 0$ and $2p^2z^2 + 2p^2 + zq > 0$.

Appendix C. A Note on Calculation

This appendix is written for those readers who wish to see certain concepts and examples in coordinates. In item **I**, we provide a program that computes the hyperbolic Monge-Ampère relative invariants S_1 and S_2 in coordinates. In items **II**, **III**, we focus on hyperbolic Euler-Lagrange systems and their types.

I. The Monge-Ampère Relative Invariants (7).

The Monge-Ampère equation (3) corresponds to the exterior differential system $(J^1(\mathbb{R}^2, \mathbb{R}), \langle \theta, \mathrm{d}\theta, \Omega \rangle_{\mathrm{alg}})$ with

$$\begin{aligned} \theta &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ \Omega &= A\mathrm{d}p \wedge \mathrm{d}q + B\mathrm{d}p \wedge \mathrm{d}y + C(\mathrm{d}x \wedge \mathrm{d}p - \mathrm{d}y \wedge \mathrm{d}q) \\ &\quad + D\mathrm{d}x \wedge \mathrm{d}q + E\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

If A, B, D, E are all zero, then the system is either empty or equivalent to the wave equation $z_{xy} = 0$. From now on, we assume that not all of A, B, D, E are zero. Under this assumption, we could always make one of the following contact transformations to arrange that $E \neq 0$:

1) $(x', y', z', p', q') = (p, q, z - px - qy, -x, -y)$;
2) $(x', y', z', p', q') = (p, y, z - px, -x, q)$;
3) $(x', y', z', p', q') = (x, q, z - qy, x, -y)$.

It then follows that, by scaling the equation, we can arrange that $E = 1$. In doing this, the hyperbolicity assumption would allow us to express

$$A = BD - C^2 + \mu^2$$

for some $\mu > 0$.

By introducing the 1-forms:

$$\begin{aligned}\eta^1 &= (C+\mu)\mathrm{d}p + D\mathrm{d}q + \mathrm{d}y,\\ \eta^2 &= -B\mathrm{d}p + (\mu - C)\mathrm{d}q - \mathrm{d}x,\\ \eta^3 &= (C-\mu)\mathrm{d}p + D\mathrm{d}q + \mathrm{d}y,\\ \eta^4 &= B\mathrm{d}p + (\mu + C)\mathrm{d}q + \mathrm{d}x,\end{aligned}$$

we obtain:

$$\Omega + \mu\mathrm{d}\theta = \eta^1 \wedge \eta^2, \qquad -\Omega + \mu\mathrm{d}\theta = \eta^3 \wedge \eta^4.$$

In other words,

$$\langle\theta, \mathrm{d}\theta, \Omega\rangle_{\mathrm{alg}} = \langle\theta, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4\rangle_{\mathrm{alg}}.$$

Letting $\eta^0 = 2\mu\theta$, we have

$$\mathrm{d}\eta^0 \equiv \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4 \mod \eta^0.$$

However, the coframing $(\eta^0, \eta^1, \ldots, \eta^4)$ is not necessarily 1-adapted (see Section 2).

Now define the coefficients $T^i_{jk} = -T^i_{kj}$ by

$$\mathrm{d}\eta^i = \frac{1}{2}T^i_{jk}\eta^j \wedge \eta^k.$$

In particular, let[12]

$$c^1 = T^1_{34}, \quad c^2 = T^2_{34}, \quad c^3 = T^3_{12}, \quad c^4 = T^4_{12}.$$

Construct a new coframing $(\omega^0, \omega^1, \ldots, \omega^4)$ by

$$\omega^0 = \eta^0, \quad \omega^i = \eta^i - c^i\eta^0, \quad (i = 1, \ldots, 4).$$

One can verify that $(\omega^0, \omega^1, \ldots, \omega^4)$ is 1-adapted; thus, it can be used to compute the Monge-Ampère relative invariants $V_1, \ldots, V_8$.

In fact, defining $\tilde{T}^i_{jk} = -\tilde{T}^i_{kj}$ using

$$\mathrm{d}\omega^i = \frac{1}{2}\tilde{T}^i_{jk}\omega^j \wedge \omega^k,$$

we obtain

$$S_1 = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{T}^1_{03} - \tilde{T}^4_{02} & \tilde{T}^1_{04} + \tilde{T}^3_{02} \\ \tilde{T}^2_{03} + \tilde{T}^4_{01} & \tilde{T}^2_{04} - \tilde{T}^3_{01} \end{pmatrix},$$

$$S_2 = \begin{pmatrix} V_5 & V_6 \\ V_7 & V_8 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{T}^1_{03} + \tilde{T}^4_{02} & \tilde{T}^1_{04} - \tilde{T}^3_{02} \\ \tilde{T}^2_{03} - \tilde{T}^4_{01} & \tilde{T}^2_{04} + \tilde{T}^3_{01} \end{pmatrix}.$$

Such expressions of V_i depend on up to the second partial derivatives of B, C, D, μ and are rather complicated; to give the reader a sense, in our calculation, $V_1, \ldots, V_8$ have 476, 159, 159, 476, 155, 262, 155 and 262 terms

[12]We remark that each c^i has 21 terms in B, C, D, μ, their partial derivatives, and p, q.

in them, respectively. We have used the following code in our calculation (with Maple™ and the `Cartan` package), where `M` stands for the function μ above.

```
> restart:
> with(Cartan):
> unprotect(D):
> Omega:= A*d(p)&^d(q) + B*d(p)&^d(y) + C*(d(x)&^d(p)- d(y)&^d(q))
  + D*d(x)&^d(q) + E*d(x)&^d(y):
> Theta:= d(x)&^d(p) + d(y)&^d(q):
> E:=1:
> A:= B*D - C^2 + M^2:
> var[1]:= x:  var[2]:= y:  var[3]:= z:  var[4]:= p:  var[5]:= q:
> for t in [B,C,D,M] do:
   d(t):= add(t[i]*d(var[i]), i = 1..5):
  od:
> for t in [B,C,D,M] do:
   for i from 1 to 5 do:
   d(t[i]):= add(t[i,j]*d(var[j]), j = 1..5):
   od:
  od:
> for t in [B,C,D,M] do:
   d(d(t)):Simf(%):ScalarForm(%):solve(%):assign(%):
  od:
> for i from 0 to 4 do:
   Form(eta[i] = 1):
  od:
> eta2coord:= {eta[0] = (d(z)- p*d(x) - q*d(y))*(2*M),
  eta[1] = (M + C)*d(p) + D*d(q) + d(y),
  eta[2] = -B*d(p) + (M - C)*d(q) - d(x),
  eta[3] = (-M + C)*d(p) + D*d(q) + d(y),
  eta[4] = B*d(p) + (M + C)*d(q) + d(x)}:
> coord2eta:= solve(eta2coord, {d(z), d(x), d(y), d(p), d(q)}):
> for i from 0 to 4 do:
   d(subs(eta2coord, eta[i])):Simf(%):subs(coord2eta,%):deta[i]:=Simf(%):
   for j from 0 to 3 do:
   for k from j+1 to 4 do:
   T[i,j,k]:= pick(deta[i], eta[j], eta[k]):
   od:
   od:
  od:
> c1:= T[1,3,4]:
  c2:= T[2,3,4]:
  c3:= T[3,1,2]:
  c4:= T[4,1,2]:
> for i from 0 to 4 do:
   Form(omega[i] = 1):
  od:
> assign(eta2coord):
> om2coord:= {omega[0] = eta[0],
  omega[1] = eta[1] - c1*eta[0],
  omega[2] = eta[2] - c2*eta[0],
  omega[3] = eta[3] - c3*eta[0],
  omega[4] = eta[4] - c4*eta[0]}:
> coord2om:= solve(om2coord, {d(z), d(x), d(y), d(p), d(q)}):
> for i from 0 to 4 do:
    d(subs(om2coord, omega[i])):Simf(%):subs(coord2om,%):  dom[i]:= Simf(%):
```

```
  for j from 0 to 3 do:
  for k from j+1 to 4 do:
  T[i,j,k]:= pick(dom[i], omega[j], omega[k]):
  od:
  od:
 od:
> V[1]:= (T[1,0,3] - T[4,0,2])/2:  V[2]:= (T[1,0,4] + T[3,0,2])/2:
 V[3]:= (T[2,0,3] + T[4,0,1])/2:  V[4]:= (T[2,0,4] - T[3,0,1])/2:
 V[5]:= (T[1,0,3] + T[4,0,2])/2:  V[6]:= (T[1,0,4] - T[3,0,2])/2:
 V[7]:= (T[2,0,3] - T[4,0,1])/2:  V[8]:= (T[2,0,4] + T[3,0,1])/2:
```

We remind the reader that S_1, S_2 are *relative* invariants under contact transformations; they transform by (8) and (9).

II. Euler-Lagrange Examples Revisited.

Given a hyperbolic Monge-Ampère PDE (3), the code above can be used to decide whether it is Euler-Lagrange and, when it *is*, its Euler-Lagrange type (positive, negative, degenerate).

For the examples 1 and 3 in Section 2, we remind the reader of their classical PDE forms; then, we continue the calculation above by computing S_2 and $\det(S_1)$ for these examples as well as for the Monge-Ampère equation satisfying (42).

1. $K = -1$ **Surfaces in** $\mathbb{E}^3$**.** (Example 1, Section 2)

Suppose that a surface $S \subset \mathbb{E}^3$ is a graph with the position vector $\mathbf{x} = (x, y, z(x, y))$. Letting $p := z_x$ and $q := z_y$, we have that, pulled back to S,

$$\theta := \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y = 0.$$

The oriented unit normal along S is just

$$\mathbf{e}_3 = (-p, -q, 1)^T(1 + p^2 + q^2)^{-1/2}.$$

Direct calculation yields the area form

$$\mathrm{d}\mathcal{A} = \frac{1}{2}(\mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}) \cdot \mathbf{e}_3 = (1 + p^2 + q^2)^{1/2}\mathrm{d}x \wedge \mathrm{d}y$$

and

$$K\mathrm{d}\mathcal{A} = \frac{1}{2}(\mathrm{d}\mathbf{e}_3 \times \mathrm{d}\mathbf{e}_3) \cdot \mathbf{e}_3 = (1 + p^2 + q^2)^{-3/2}\mathrm{d}p \wedge \mathrm{d}q,$$

where K is the Gauss curvature of S.

It follows that the system for $K = -1$ is characterized by

$$\theta = 0, \qquad \mathrm{d}p \wedge \mathrm{d}q + (1 + p^2 + q^2)^2\mathrm{d}x \wedge \mathrm{d}y = 0.$$

Noting that $\mathrm{d}x \wedge \mathrm{d}y \neq 0$ on S, the PDE form of this system is:

$$(z_{xx}z_{yy} - z_{xy}^2) + (1 + z_x^2 + z_y^2)^2 = 0. \tag{43}$$

2. $K = 1$ **Time-like Surfaces in** $\mathbb{E}^{2,1}$**.** (Example 3, Section 2)

Since we use the signature $(+,-,+)$ for the Lorentzian metric on $\mathbb{E}^{2,1}$, the dot and cross products are defined, respectively, by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 - u_2v_2 + u_3v_3, \qquad \mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_1v_3 - u_3v_1 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

Let S be a time-like surface with the position vector $\mathbf{x}$ and a pseudo-orthonormal frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ attached to it ($\mathbf{e}_1, \mathbf{e}_2$ being tangent to S). The area form $\mathrm{d}\mathcal{A}$ and the Gauss curvature K can be computed by

$$\mathrm{d}\mathcal{A} = \frac{1}{2}(\mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{x})\cdot\mathbf{e}_3,$$

and

$$K\mathrm{d}\mathcal{A} = \frac{1}{2}(\mathrm{d}\mathbf{e}_3\times\mathrm{d}\mathbf{e}_3)\cdot\mathbf{e}_3,$$

where the 2-forms on the right-hand-sides are their pull-backs to S.

When S is a graph $\mathbf{x} = (x, y, z(x,y))$, it is easy to find that

$$\mathbf{e}_3 = (-p, q, 1)^T(1 + p^2 - q^2)^{-1/2}.$$

Direct calculation yields that

$$\mathrm{d}\mathcal{A} = (1 + p^2 - q^2)^{1/2}\mathrm{d}x\wedge\mathrm{d}y, \quad K\mathrm{d}\mathcal{A} = -(1 + p^2 - q^2)^{-3/2}\mathrm{d}p\wedge\mathrm{d}q.$$

Therefore, the equation for $K = 1$ is

$$(z_{xx}z_{yy} - z_{xy}^2) + (1 + z_x^2 - z_y^2)^2 = 0. \tag{44}$$

Now, calculating using Maple™, we find:[13]

Eq.	S_2	$\det(S_1)$	E-L Type
(43)	$\mathbf{0}$	$-\dfrac{1}{16}(p^2 + q^2 + 1)$	$-$
(44)	$\mathbf{0}$	$\dfrac{1}{16}(p^2 - q^2 + 1)$	$+$
(42)	$\mathbf{0}$	$\dfrac{z^2q^4(4p^2z^5 - 16p^2z^3 + qz^4 - 20p^2z - 8qz^2 - q)^2}{32(2p^2z^2 + 2p^2 + zq)^3(z^2+1)^6}$	$+$

The Euler-Lagrange types are consistent with our earlier observation when calculation was made using differential forms. Here, we have used the following code for computing $\det(S_1)$ and S_2 associated to (42), which can be easily modified to work for (43) and (44).

```
> ABCDE:= {eA = 2*q*z*(z∧2+1)∧3,
   eB = 2*q∧2*(z∧2+1)∧2*(4*p∧2*z∧3 - q*z∧2 + 4*p∧2*z + 3*q),
```

[13]Note that, in (42), $2p^2z^2 + 2p^2 + zq > 0$, by hyperbolicity.

```
   eC = -2*p*q*(z^2+1)^3*(4*p^2*z+q),
   eD = (z^2+1)*(4*p^2*z^3+q*z^2+4*p^2*z - q)*(2*p^2*z^2+q*z+2*p^2),
   eE = -q^3*(4*p^2*z^5+q*z^4 - 16*p^2*z^3 - 8*q*z^2 - 20*p^2*z - q)}:
> BCDM:= subs(ABCDE, {B = eB/eE, C = eC/eE, D = eD/eE,
   M = sqrt((eA*eE + eC^2 - eB*eD))/eE}):
> dBCDM:= {}:  ddBCDM:= {}:
> for t in [B,C,D,M] do:
   for i from 1 to 5 do:
   dBCDM:= {op(dBCDM), t[i] = diff(subs(BCDM, t), var[i])}:
   od:
  od:
> for t in [B,C,D,M] do:
   for i from 1 to 5 do:
   for j from 1 to 5 do:
   ddBCDM:= {op(ddBCDM), t[i,j] = diff(diff(subs(BCDM, t), var[i]), var[j])}:
   od:
   od:
  od:
> for i from 5 to 8 do:
   simplify(subs(BCDM union dBCDM union ddBCDM, V[i]));
  od;
> subs(BCDM union dBCDM union ddBCDM, V[1]*V[4]-V[2]*V[3]):simplify(%)
```

III. The Lagrangian.

Classically, a central object in a 2D variational problem is the functional:

$$\int_{\mathcal{D}} L(x, y, z, z_x, z_y)\mathrm{d}x\mathrm{d}y,$$

where L is the Lagrangian (function) and $\mathcal{D} \subset \mathbb{R}^2$ a fixed compact domain.

The Euler-Lagrange equation associated to a fixed-boundary variation is

$$L_z = \frac{\mathrm{d}L_{z_x}}{\mathrm{d}x} + \frac{\mathrm{d}L_{z_y}}{\mathrm{d}y}. \tag{45}$$

A coordinate-independent formulation (see [BGG03, Section 1.2]) considers instead a fixed boundary variation in $J^1(\mathbb{R}^2, \mathbb{R})$ (with standard coordinates (x, y, z, p, q)) by Legendre surfaces[14] S and the functional:

$$\int_S \Lambda,$$

where $\Lambda = L(x, y, z, p, q)\mathrm{d}x \wedge \mathrm{d}y$. This variational problem may seem less restrictive than the classical one,[15] but the corresponding Euler-Lagrange equation is the same as (45), which is Monge-Ampère (3) with

$$A = 0, \quad B = L_{pp}, \quad C = L_{pq}, \quad D = L_{qq},$$
$$E = L_{xp} + pL_{zp} + L_{yq} + qL_{zq} - L_z.$$

More generally but still in the 2D case, Λ can be any 2-form on $J^1(\mathbb{R}^2, \mathbb{R})$. The 'stationary point' of a corresponding fixed-boundary variational problem is characterized by a more general Euler-Lagrange system (see [BGG03,

[14]That is, surfaces that annihilate the pull-back of $\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$.

[15]Note that a fixed-boundary variation in the classical formulation may not lift to be a fixed-boundary variation by Legendre surfaces in $J^1(\mathbb{R}^2, \mathbb{R})$.

Section 1.2.2]). It is this latter notion that we have in mind when we refer to Euler-Lagrange systems.

Theoretically, the inverse problem:

> *Given a hyperbolic Monge-Ampère system with $S_2 = \mathbf{0}$, can one construct an associated Lagrangian 2-form Λ in some local coordinates?*

is solved in [BGG03, Theorems 1.2 and 2.2]. Here we only outline the steps of a construction and present some examples.

Start with a hyperbolic Monge-Ampère system with $S_2 = \mathbf{0}$.

Step 1. Find a 1-adapted coframing $\boldsymbol{\omega} = (\omega^0, \ldots, \omega^4)$;
Step 2. Compute the 1-form ϕ_0 in (4), which is determined, using $\boldsymbol{\omega}$;
Step 3. Since, $S_2 = \mathbf{0}$, ϕ_0 must be closed; thus, find a function λ such that $\mathrm{d}\lambda = 2\lambda\phi_0$;
Step 4. It would then follow that $\Pi := \lambda\omega^0 \wedge (\omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4)$ is closed; hence, it is a Poincaré-Cartan form. Now find[16] a 2-form Λ such that $\mathrm{d}\Lambda = \Pi$. This Λ is a desired Lagrangian.

Now, for (43), (44) and the Monge-Ampère equation satisfying (42), we follow these steps to compute their representative Lagrangian 2-forms and summarize intermediate expressions in the following tables.

Eq.	(43)
ϕ_0	$\dfrac{p\mathrm{d}p + q\mathrm{d}q}{1+p^2+q^2}$
λ	$1+p^2+q^2$
Π	$4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \dfrac{4(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y)}{(1+p^2+q^2)^2}\mathrm{d}p \wedge \mathrm{d}q$
Λ	$4z\ \mathrm{d}x \wedge \mathrm{d}y + \dfrac{4z}{(1+p^2+q^2)^2}\mathrm{d}p \wedge \mathrm{d}q - \dfrac{2}{1+p^2+q^2}(\mathrm{d}x \wedge \mathrm{d}q - \mathrm{d}y \wedge \mathrm{d}p)$

Eq.	(44)
ϕ_0	$\dfrac{p\mathrm{d}p - q\mathrm{d}q}{1+p^2-q^2}$
λ	$1+p^2-q^2$
Π	$4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \dfrac{4(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y)}{(1+p^2-q^2)^2}\mathrm{d}p \wedge \mathrm{d}q$
Λ	$4z\ \mathrm{d}x \wedge \mathrm{d}y + \dfrac{4z}{(1+p^2-q^2)^2}\mathrm{d}p \wedge \mathrm{d}q - \dfrac{2}{1+p^2-q^2}(\mathrm{d}x \wedge \mathrm{d}q + \mathrm{d}y \wedge \mathrm{d}p)$

For the Monge-Ampère equation satisfying (42), we record only λ and Λ, since the expressions of ϕ_0 and Π are rather complicated.

[16]This step can involve using the proof of the Poincaré Lemma.

Eq.	Monge-Ampère Equation (3) satisfying (42)
λ	$\dfrac{q^2(4p^2z^5-16p^2z^3+qz^4-20p^2z-8qz^2-q)^2}{(2p^2z^2+2p^2+qz)^2(z^2+1)^4}$
$\dfrac{(2z^2p^2+2p^2+qz)^{1/2}}{8\sqrt{2}}\Lambda$	$4(z^2+1)\mathrm{d}y\wedge\mathrm{d}p+\dfrac{(px+2qy)(8p^2z+2q)(z^2+1)}{q(2z^2p^2+2p^2+qz)}\mathrm{d}z\wedge\mathrm{d}p$ $-\dfrac{z}{q}\mathrm{d}x\wedge\mathrm{d}q+\dfrac{2q(z^2-1)}{z^2+1}\mathrm{d}x\wedge\mathrm{d}y+\Sigma\,\mathrm{d}z\wedge\mathrm{d}q$

In the above,

$$\begin{aligned}\Sigma=\frac{1}{2(z^2+1)q^2(2p^2z^2+2p^2+zq)}\Big(&-8p^3z^6+(-16p^4x-16p^3qy-4pq)z^5\\&+(-12p^2qx-4pq^2y-24p^3)z^4+(-32p^4x-32p^3qy-3q^2x-8pq)z^3\\&+(-12p^2qx-8pq^2y-24p^3)z^2+(-16p^4x-16p^3qy+q^2x-4pq)z\\&-4pq^2y-8p^3\Big).\end{aligned}$$

Finally, we remind the reader that Λ is not unique in the sense that one can add to it any exact 2-form and any 2-form in the contact ideal generated by $\mathrm{d}z-p\mathrm{d}x-q\mathrm{d}y$. For details, see [BGG03, Section 1.1].

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