GEOMETRY OF BÄCKLUND TRANSFORMATIONS I: GENERALITY

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Abstract. Using Élie Cartan's method of equivalence, we prove an upper bound for the generality of generic rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. In cases when the Bäcklund transformation admits a symmetry group whose orbits have codimension 1, 2 or 3, we obtain classification results and new examples of auto-Bäcklund transformations.

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1. Introduction

In 1882, the Swedish mathematician Albert V. Bäcklund proved the result (see [Bäc83], [BGG03] or [CT80]): *Given a surface with a constant Gauss curvature $K < 0$ in $\mathbb{E}^3$, one can construct, by solving ODEs, a 1-parameter family of new surfaces in $\mathbb{E}^3$ with the Gauss curvature K.* This is the origin of the term "Bäcklund transformation".

Classically, a *Bäcklund transformation* is a PDE system $\mathcal{B}$ that relates solutions of two other PDE systems $\mathcal{E}_1$ and $\mathcal{E}_2$. Moreover, such a relation must satisfy the property: given a solution u of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$), substituting it in $\mathcal{B}$, one would obtain a PDE system whose solutions can be found by

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ODE methods and produce solutions of $\mathcal{E}_2$ (resp. $\mathcal{E}_1$). If, in addition, $\mathcal{E}_1$ and $\mathcal{E}_2$ are contact equivalent to each other, then the corresponding Bäcklund transformation is called an *auto-Bäcklund transformation*.

For example, the *Cauchy-Riemann system*

$$\begin{cases} u_x - v_y = 0, \\ u_y + v_x = 0 \end{cases} \tag{1}$$

is an auto-Bäcklund transformation; it relates solutions of the Laplace equation $\Delta z = 0$ for $z(x, y)$ in the following way: If u satisfies $\Delta u = 0$, then, substituting it in (1), one obtains a compatible first-order system for v, whose solutions can be found by ODE methods and satisfy $\Delta v = 0$, and *vice versa*.

As another example, consider the system of nonlinear equations

$$\begin{cases} z_x - \bar{z}_x = \lambda \sin(z + \bar{z}), \\ z_y + \bar{z}_y = \lambda^{-1} \sin(z - \bar{z}), \end{cases} \tag{2}$$

where λ is a nonzero constant. One can easily verify that (2) is an auto-Bäcklund transformation relating solutions of the *sine-Gordon equation*

$$u_{xy} = \frac{1}{2}\sin(2u). \tag{3}$$

The system (2) can be derived from the classical auto-Bäcklund transformation relating surfaces in $\mathbb{E}^3$ with a negative constant Gauss curvature. For details, see [CT80].

In general, there may seem to be very few restrictions on the types of PDE systems that admit a Bäcklund transformation. In addition to the elliptic and hyperbolic examples mentioned above, a Bäcklund transformation may exist relating solutions of a parabolic equation (see [NC82]) or an equation of order higher than 2, for example, the KdV equation (see [WE73]). Furthermore, two PDE systems being Bäcklund-related need not be contact equivalent to each other (see [CI09]).

The importance of Bäcklund transformations may, in part, be viewed through their relation to surface geometry and mathematical physics. On the geometry side, Bäcklund transformations allow one to obtain new surfaces with prescribed geometric properties from old. For a variety of such examples, see [RS02]. On the mathematical physics side, a prototypical result is that the Bäcklund transformation (2), when applied to the trivial solution $z(x, y) = 0$ of (3), yields a 2-parameter family of 1-soliton solutions of the sine-Gordon equation (see [TU00]). More elaborate techniques have since been developed to find the so-called multi-soliton solutions of nonlinear PDE systems (for example, the KdV equation), using Bäcklund transformations.

An ultimate goal of studying Bäcklund transformations is solving the *Bäcklund problem*, which was considered by É. Goursat in [Gou25]:

Find all pairs of systems of PDEs whose solutions are related by a Bäcklund transformation.

Although this problem remains largely unsolved, recent works of Clelland and Ivey ([Cle02],[CI05] and [CI09]) and those of Anderson and Fels ([AF12], [AF15]) have pointed out new directions for studying Bäcklund transformations. Instead of aiming at constructing new examples or finding techniques of calculating explicit solutions to PDE systems, they work in a geometric setting that is natural to the study of structural properties (of Bäcklund transformations) that are invariant under contact transformations. Under such settings, a complete classification of Bäcklund transformations, at least in certain cases, is possible by using É. Cartan's method of equivalence.

The current work is concerned with the geometric aspect of Bäcklund transformations, not so much in the sense of relating to classical surface geometry, as in that of seeing Bäcklund transformations as geometric objects and studying their invariants.

More specifically, we study nontrivial rank-1 Bäcklund transformations (see Definition 2.10) relating a pair of hyperbolic Monge-Ampère systems. Since many classical examples belong to this category, it is highly desirable to have a complete classification of Bäcklund transformations of this kind. In [Cle02], by establishing a G-structure associated to a Bäcklund transformation, Clelland approached the classification problem using Cartan's method of equivalence, restricting to the case when all local invariants of the structure are constants (a.k.a the homogeneous case). Her classification found 15 types, within which 11 are analogues of the classical Bäcklund transformation between surfaces in $\mathbb{E}^3$ with a negative constant Gauss curvature.

Since homogeneous structures, up to equivalence, depend only on constants, the following question remains to be answered: *What kind of initial data do we need to specify in order to determine a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems?*

In Section 5, in the generic case, we use the method of equivalence (see [Gar89], [Bry14]) to prove an upper bound for the magnitude of such initial data:

To determine a generic Bäcklund transformation relating two hyperbolic Monge-Ampère systems, it is sufficient to specify at most 6 *functions of* 3 *variables.*

It is an immediate consequence of our theorem that most hyperbolic Monge-Ampère systems are not related to any system of the same type by a generic Bäcklund transformation.

A major difficulty in using the method of equivalence to classify Bäcklund transformations lies in verifying the compatibility of large systems of polynomial equations, whose variables are the Bäcklund structure invariants and their covariant derivatives. However, we found that such calculation becomes much more manageable when we set two structure invariants to be specific constants (see Section 6). The corresponding Bäcklund transformations are either homogeneous (corresponding to a case already classified in [Cle02]) or of cohomogeneity 1, 2 or 3. In the cohomogeneity-1 case, we obtain an auto-Bäcklund transformation of a homogeneous Euler-Lagrange system that is contact equivalent to the equation

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} = 0, \tag{29}$$

where $A = 2z_x + y$ and $B = 2z_y - x$. The cohomogeneity-2 case has a subcase that arises when the 'free derivatives' associated to the structure are expressed in terms of the primary invariants. In this subcase, the corresponding Lie algebra of symmetry must be of the form $\mathfrak{q} \oplus \mathbb{R}$ where $\mathfrak{q}$ is either $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{so}(3, \mathbb{R})$ or the solvable 3-dimensional Lie algebra with basis $\{x_i\}_{i=1}^3$ satisfying

$$[x_2, x_3] = x_1, \quad [x_3, x_1] = x_2, \quad [x_1, x_2] = 0.$$

In particular, when $\mathfrak{q}$ is solvable, we obtain an auto-Bäcklund transformation; the underlying Monge-Ampère system is Euler-Lagrange, of cohomogeneity 1, and contact equivalent to the equation

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} + (A^2 + B^2)^2 = 0, \tag{33}$$

where $A = z_x - y$, $B = z_y + x$.

2. Definitions and Notations

In this section, we present some definitions and notations to be used later.

2.1. **Exterior Differential Systems** (c.f. [BCG+13]).

Definition 2.1. Let M be a smooth manifold, $\mathcal{I} \subset \Omega^*(M)$ a graded ideal that is closed under exterior differentiation. The pair $(M, \mathcal{I})$ is said to be an *exterior differential system* with space M and differential ideal $\mathcal{I}$.

Given an exterior differential system $(M, \mathcal{I})$, we use $\mathcal{I}^k$ to denote the degree-k piece of $\mathcal{I}$, namely, $\mathcal{I}^k = \mathcal{I} \cap \Omega^k(M)$, where $\Omega^k(M)$ stands for the $C^\infty(M)$-module of differential k-forms on M. If the rank of $\mathcal{I}^k$, restricted to each point, is locally a constant, then the elements of $\mathcal{I}^k$ are precisely smooth sections of a vector bundle denoted by I^k.

Definition 2.2. An *integral manifold* of an exterior differential system $(M, \mathcal{I})$ is an immersed submanifold $i : N \hookrightarrow M$ satisfying $i^*\phi = 0$ for any $\phi \in \mathcal{I}$.

Intuitively, an exterior differential system is a coordinate-independent way to express a PDE system; an integral manifold, usually with a certain independence condition satisfied, corresponds to a solution of the PDE system.

Definition 2.3. Two exterior differential systems $(M, \mathcal{I})$ and $(N, \mathcal{J})$ are said to be *equivalent up to diffeomorphism*, or *equivalent*, for brevity, if there exists a diffeomorphism $\phi : M \to N$ such that $\phi^*\mathcal{J} = \mathcal{I}$. Such a ϕ

is called an *equivalence* between the two systems. An equivalence between $(M, \mathcal{I})$ and itself is called a *symmetry* of $(M, \mathcal{I})$.

Definition 2.4. Let $\pi : N \to M$ be a submersion. A p-form $\omega \in \Omega^p(N)$ is said to be π-*semi-basic* if, for any $x \in N$, $\omega|_x \in \pi^*(\Lambda^p(T^*M))$.

Definition 2.5. Let M be a smooth manifold. Let $E \subset \Lambda^k(T^*M)$ be a vector subbundle, and X a smooth vector field defined on M. We say that E is *invariant under the flow of* X if, for any (smooth) local section $\omega : U \to E$, where $U \subset M$ is open, the Lie derivative $\mathcal{L}_X\omega$ remains a section of E (over U).

Notation 1. Let $[\![\theta_1, \dots, \theta_\ell]\!]$ denote the vector subbundle of $\Lambda^k(T^*U)$ generated by differential forms $\theta_1, \dots, \theta_\ell$ (defined on U, an open subset of a smooth manifold) of the same degree k.

2.2. **Hyperbolic Monge-Ampère Systems** (c.f. [BGG03]).

Among second order PDEs for 1 unknown function of 2 independent variables, *Monge-Ampère equations* are those of the form

$$A(z_{xx}z_{yy} - z_{xy}^2) + Bz_{xx} + 2Cz_{xy} + Dz_{yy} + E = 0, \tag{4}$$

where A, B, C, D, E are functions of x, y, z, z_x, z_y. A Monge-Ampère equation (4) is said to be *elliptic* (resp., *hyperbolic*, *parabolic*) if $AE - BD + C^2$ is negative (resp., positive, zero).

A Monge-Ampère equation can be formulated as an exterior differential system on a contact manifold. In the hyperbolic case, we follow [BGH95] to give the following definition.

Definition 2.6. A *hyperbolic Monge-Ampère system* $(M, \mathcal{I})$ is an exterior differential system, where M is a 5-manifold, $\mathcal{I}$ being locally algebraically generated by $\theta \in \mathcal{I}^1$ and $\mathrm{d}\theta, \Omega \in \mathcal{I}^2$ satisfying

(1) $\theta \wedge (\mathrm{d}\theta)^2 \neq 0$;
(2) $[\![\mathrm{d}\theta, \Omega]\!]$, modulo θ, has rank 2;
(3) $(\lambda \mathrm{d}\theta + \mu\Omega)^2 \equiv 0 \mod \theta$ has two distinct solutions $[\lambda_i : \mu_i] \in \mathbb{RP}^1$ $(i = 1, 2)$.

Here, condition (3), in particular, characterizes hyperbolicity: Each integral surface of $(M, \mathcal{I})$ is foliated by two distinct families of characteristics.

Definition 2.7. Consider a hyperbolic Monge-Ampère system $(M, \mathcal{I})$. A local coframing $\boldsymbol{\theta} = (\theta^0, \theta^1, \dots, \theta^4)$ defined on an open neighborhood $U \subset M$ is said to be 0-*adapted* if, on U,

$$\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle.$$

The condition for 0-adaptedness as defined above is a pointwise condition on $\boldsymbol{\theta}$. In fact, any 0-adapted coframing associated to $(M, \mathcal{I})$ is a local section of a G_0-structure $\mathcal{G}_0$ on M, where $G_0 \subset \mathrm{GL}(5,\mathbb{R})$ is the subgroup generated by matrices of the form

$$g = \begin{pmatrix} a & \mathbf{0} & \mathbf{0} \\ \mathbf{b}_1 & A & 0 \\ \mathbf{b}_2 & 0 & B \end{pmatrix}, \qquad a \neq 0;\ A, B \in \mathrm{GL}(2,\mathbb{R});\ \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2,$$

and

$$J = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & I_2 \\ \mathbf{0} & I_2 & 0 \end{pmatrix}.$$

Two hyperbolic Monge-Ampère systems are equivalent if and only if their corresponding G_0-structures are equivalent.[1]

2.3. **Integrable Extensions and Bäcklund Transformations** (c.f. [AF12], [AF15]).

Definition 2.8. Let $(M, \mathcal{I})$ be an exterior differential system. A *rank-k integrable extension* of $(M, \mathcal{I})$ is an exterior differential system $(N, \mathcal{J})$ with a submersion $\pi : N \to M$ that satisfies the condition: for each $p \in N$, there exists an open neighborhood $U \subset N$ $(p \in U)$ such that

(1) on U, the differential ideal $\mathcal{J}$ is algebraically generated by the elements of $\pi^*\mathcal{I}$ together with 1-forms $\theta_1, \ldots, \theta_k \in \Omega^1(U)$, where $k = \dim N - \dim M$;
(2) for any $p \in U$, let F_p denote the fiber $\pi^{-1}(\pi(p))$; the 1-forms $\theta_1, \ldots, \theta_k$ restrict to T_pF_p to be linearly independent.

Remark 1. In Definition 2.8, one can understand $\mathcal{J}$ as defining a connection on the bundle $\pi : N \to M$ that is flat over the integral manifolds of $\mathcal{I}$. More specifically, Condition (1) implies that, if $S \subset M$ is an integral manifold of $(M, \mathcal{I})$, then $\mathcal{J}$ restricts to $\pi^{-1}(S)$ to be Frobenius; hence, locally, $\pi^{-1}(S)$ is foliated by integral manifolds of $(N, \mathcal{J})$. Condition (2) implies that, restricting to any integral manifold of $(N, \mathcal{J})$, π is an immersion, whose image is an integral manifold of $(M, \mathcal{I})$.

Definition 2.9. A *Bäcklund transformation* relating two exterior differential systems, $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$, is a quadruple $(N, \mathcal{B}; \pi_1, \pi_2)$ where, for each $i \in \{1, 2\}$, $\pi_i : N \to M_i$ makes $(N, \mathcal{B})$ an integrable extension of $(M_i, \mathcal{I}_i)$. Such a Bäcklund transformation is represented by the diagram:

$$\begin{array}{ccc} & (N, \mathcal{B}) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (M_1, \mathcal{I}_1) & & (M_2, \mathcal{I}_2) \end{array}$$

[1]Two G-structures $\mathcal{G}$ and $\hat{\mathcal{G}}$ on a manifold M are said to be *equivalent* if there exists a diffeomorphism $\phi : \mathcal{G} \to \hat{\mathcal{G}}$ such that $\phi^*\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}$, where $\boldsymbol{\omega}, \hat{\boldsymbol{\omega}}$ are the tautological 1-forms on $\mathcal{G}, \hat{\mathcal{G}}$, respectively.

Definition 2.10. In Definition 2.9, if M_1, M_2 have the same dimension, which is not required in general, then we define the *rank* of $(N, \mathcal{B}; \pi_1, \pi_2)$ to be the fiber dimension of either π_1 or π_2. If $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$ are equivalent exterior differential systems, then $(N, \mathcal{B}; \pi_1, \pi_2)$ is called an *auto-Bäcklund transformation* of either $(M_i, \mathcal{I}_i)$.

Example 1. Let $(N, \mathcal{B}; \pi, \bar{\pi})$ be a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$. On some open subsets $U \subset M$ and $\bar{U} \subset \bar{M}$, we can choose 0-adapted coframings such that

$$\mathcal{I} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle, \qquad \bar{\mathcal{I}} = \langle \bar{\eta}^0, \bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4 \rangle.$$

Let $V = \pi^{-1}U \cap \bar{\pi}^{-1}\bar{U}$, assumed to be nonempty. It is easy to see that the Cauchy characteristics of the system $\langle \pi^* \eta^0 \rangle$ (defined on V) are precisely the fibers of $\pi|_V$; similarly for $\bar{\pi}|_V$. Thus, it is natural to regard $(N, \mathcal{B}; \pi, \bar{\pi})$ as nontrivial if $\pi^*\eta^0$ and $\bar{\pi}^*\bar{\eta}^0$ are linearly independent 1-forms on N. In particular, it follows that, on V, the differential ideal $\mathcal{B}$ is algebraically generated by $\pi^*\mathcal{I}$ and $\bar{\pi}^*\bar{\eta}^0$ as well as by $\bar{\pi}^*\bar{\mathcal{I}}$ and $\pi^*\eta^0$.

Definition 2.11. Given a fiber bundle $\pi : E \to B$, for any $p \in E$, the *vertical tangent space* of E at p is by definition the kernel of $\pi_* : T_pE \to T_{\pi(p)}B$.

Definition 2.12. A Bäcklund transformation $(N, \mathcal{B}; \pi_1, \pi_2)$ is said to be *nontrivial* if the two fibrations π_1, π_2 have distinct vertical tangent spaces at each point $p \in N$.

3. Monge-Ampère Systems and Their First Invariants

Let $(M, \mathcal{I})$ be a hyperbolic Monge-Ampère system. Let $\mathcal{G}_0$ denote the G_0-structure on $(M, \mathcal{I})$ (see Definition 2.7). One can reduce (see [BGG03]) $\mathcal{G}_0$ to a G_1-structure $\mathcal{G}_1$ on which the tautological 1-forms $\omega^0, \omega^1, \ldots, \omega^4$ satisfy the following structure equations:

$$(5) \qquad \mathrm{d}\begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = -\begin{pmatrix} \phi_0 & 0 & 0 & 0 & 0 \\ 0 & \phi_1 & \phi_2 & 0 & 0 \\ 0 & \phi_3 & \phi_4 & 0 & 0 \\ 0 & 0 & 0 & \phi_5 & \phi_6 \\ 0 & 0 & 0 & \phi_7 & \phi_8 \end{pmatrix} \wedge \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \\ (V_1 + V_5)\omega^0 \wedge \omega^3 + (V_2 + V_6)\omega^0 \wedge \omega^4 \\ (V_3 + V_7)\omega^0 \wedge \omega^3 + (V_4 + V_8)\omega^0 \wedge \omega^4 \\ (V_8 - V_4)\omega^0 \wedge \omega^1 + (V_2 - V_6)\omega^0 \wedge \omega^2 \\ (V_3 - V_7)\omega^0 \wedge \omega^1 + (V_5 - V_1)\omega^0 \wedge \omega^2 \end{pmatrix},$$

where $\phi_0 = \phi_1 + \phi_4 = \phi_5 + \phi_8$, and $G_1 \subset G_0$ is the subgroup generated by

$$(6) \qquad g = \begin{pmatrix} a & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & 0 \\ \mathbf{0} & 0 & B \end{pmatrix}, \ A, B \in \mathrm{GL}(2, \mathbb{R}), \ a = \det(A) = \det(B),$$

and

$$J = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & I_2 \\ \mathbf{0} & I_2 & 0 \end{pmatrix} \in \mathrm{GL}(5, \mathbb{R}). \tag{7}$$

Definition 3.1. Let $(M, \mathcal{I})$ be a hyperbolic Monge-Ampère system. A *1-adapted coframing*[2] of $(M, \mathcal{I})$ with domain $U \subset M$ is a section $\boldsymbol{\eta} : U \to \mathcal{G}_1$.

Following [BGG03], we introduce the notation[3]

$$S_1 := \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}, \quad S_2 := \begin{pmatrix} V_5 & V_6 \\ V_7 & V_8 \end{pmatrix}. \tag{8}$$

It is shown in [BGG03] that

Proposition 3.1. *Along each fiber of $\mathcal{G}_1$,*

$$S_i(u \cdot g) = aA^{-1} S_i(u) B, \quad (i = 1, 2) \tag{9}$$

for any $g = \mathrm{diag}(a; A; B)$ in the identity component of G_1. Moreover,

$$S_1(u \cdot J) = \begin{pmatrix} -V_4 & V_2 \\ V_3 & -V_1 \end{pmatrix}, \quad S_2(u \cdot J) = \begin{pmatrix} V_8 & -V_6 \\ -V_7 & V_5 \end{pmatrix}. \tag{10}$$

Proposition 3.1 has a simple interpretation: the matrices S_1 and S_2 correspond to two invariant tensors under the G_1-action. In fact, one can verify that the quadratic form

$$\Sigma_1 := V_3\, \omega^1 \omega^3 - V_1\, \omega^1 \omega^4 + V_4\, \omega^2 \omega^3 - V_2\, \omega^2 \omega^4 \tag{11}$$

and the 2-form

$$\Sigma_2 := V_7\, \omega^1 \wedge \omega^3 - V_5\, \omega^2 \wedge \omega^3 + V_8\, \omega^1 \wedge \omega^4 - V_6\, \omega^2 \wedge \omega^4 \tag{12}$$

are G_1-invariant, which implies that Σ_1, Σ_2 are locally well-defined on $(M, \mathcal{I})$.

An infinitesimal version of Proposition 3.1 will be useful: for $i = 1, 2$,

$$\mathrm{d}S_i \equiv \begin{pmatrix} \phi_4 & -\phi_2 \\ -\phi_3 & \phi_1 \end{pmatrix} S_i + S_i \begin{pmatrix} \phi_5 & \phi_6 \\ \phi_7 & \phi_8 \end{pmatrix} \mod \omega^0, \omega^1, \ldots, \omega^4. \tag{13}$$

An important class of Monge-Ampère systems are the *Euler-Lagrange systems*. In the classical calculus of variations, an Euler-Lagrange system is a PDE system whose solutions correspond to the stationary points of a given first-order functional. In [BGG03], it is shown:

Proposition 3.2. ([BGG03]) *A hyperbolic Monge-Ampère system is locally equivalent to an Euler-Lagrange system if and only if S_2 vanishes.*

Remark 2. Proposition 3.2 says that the property of being *Euler-Lagrange* is intrinsically defined, that is, it does not depend on the choice of local coordinates.

[2]This is not to be confused with a 1-adapted coframing in the sense of Definition 5.1.

[3]These S_i are those defined in [BGG03] with the same notation scaled by 1/2.

4. G-structure Equations for Bäcklund Transformations

In Definition 2.9, it may appear that $(N, \mathcal{B}; \pi_1, \pi_2)$ being a Bäcklund transformation imposes conditions on all components in this quadruple. However, when it is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems, one only needs to impose conditions on the exterior differential system[4] $(N, \mathcal{B})$, as the following proposition shows.

Proposition 4.1. [Cle02] *An exterior differential system $(N^6, \mathcal{B})$ is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems if and only if, for each $p \in N$, there exists an open neighborhood $V \subset N$ $(p \in V)$, a coframing $(\theta^0, \bar\theta^0, \theta^1, \dots, \theta^4)$ and nonvanishing functions $A_1, \dots, A_4$ $(A_1A_4 \neq A_2A_3)$ defined on V, satisfying the conditions:*

(1) *the differential ideal $\mathcal{B} = \langle \theta^0, \bar\theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle_{\text{alg}}$;*
(2) *the vector bundles $E_0 = [\![\theta^0]\!]$, $E_1 = [\![\theta^0, \theta^1, \theta^2]\!]$ and $E_2 = [\![\theta^0, \theta^3, \theta^4]\!]$ are invariant along the flow of X (see Definition 2.5), where X is a nonvanishing vector field on V that annihilates $\theta^0, \theta^1, \dots, \theta^4$;*
($\bar{2}$) *the vector bundles $\bar E_0 = [\![\bar\theta^0]\!]$, $\bar E_1 = [\![\bar\theta^0, \theta^1, \theta^2]\!]$ and $\bar E_2 = [\![\bar\theta^0, \theta^3, \theta^4]\!]$ are invariant along the flow of $\bar X$, where $\bar X$ is a nonvanishing vector field on V that annihilates $\bar\theta^0, \theta^1, \dots, \theta^4$;*
(3) *the following congruences hold:*

$$\mathrm{d}\theta^0 \equiv A_1\theta^1 \wedge \theta^2 + A_2\theta^3 \wedge \theta^4 \mod \theta^0,$$
$$\mathrm{d}\bar\theta^0 \equiv A_3\theta^1 \wedge \theta^2 + A_4\theta^3 \wedge \theta^4 \mod \bar\theta^0.$$

This proposition has the following corollary.

Corollary 4.1. *Let $(N^6, \mathcal{B}; \pi_1, \pi_2)$ be a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. A coframing defined on an open subset $V \subset N$ that satisfies Conditions (1)-(3) in Proposition 4.1 can always be arranged to satisfy the extra condition: $A_2 = A_3 = 1$.*

Proof. This is obtained by scaling θ^0 and $\bar\theta^0$. □

Definition 4.1. A coframing as concluded in Corollary 4.1 is said to be 0-*adapted* to the Bäcklund transformation $(N, \mathcal{B})$.

Given a nontrivial rank-1 Bäcklund transformation $(N, \mathcal{B}; \pi_1, \pi_2)$ relating two hyperbolic Monge-Ampère systems, one can ask whether its 0-adapted coframings are precisely the local sections of a G-structure on N. However, this is not true. For example, consider a 0-adapted coframing $(\theta^0, \bar\theta^0, \theta^1, \dots, \theta^4)$ defined on an open subset $U \subset N$ with corresponding functions A_1, A_4. Let $T : U \to \mathrm{GL}(6, \mathbb{R})$ be the transformation:

$$T(p) : (\theta^0, \bar\theta^0, \theta^1, \theta^2, \theta^3, \theta^4) \mapsto \left(\frac{1}{A_1}\theta^0, \frac{1}{A_4}\bar\theta^0, \theta^3, \theta^4, \theta^1, \theta^2\right), \quad \forall p \in U. \tag{14}$$

[4] To be more precise, $(N, \mathcal{B})$ is a hyperbolic exterior differential system of type $s = 2$ in the sense of [BGH95].

It is easy to see that the coframing on the right-hand-side is 0-adapted. However, the same transformation, when applied to a 0-adapted coframing with corresponding functions A_1', A_4' that are different from A_1, A_4, may not result in a 0-adapted coframing.

One simple strategy, as taken by [Cle02], to avoid this issue is by, in addition to understanding the subbundles $[\![\theta^0]\!]$ and $[\![\bar{\theta}^0]\!]$ as an ordered pair, fixing an order for the pair of subbundles $[\![\theta^0, \bar{\theta}^0, \theta^1, \theta^2]\!]$ and $[\![\theta^0, \bar{\theta}^0, \theta^3, \theta^4]\!]$. Once this is considered, all local 0-adapted coframings respecting such an ordering are precisely the local sections of a G-structure, where $G \subset \mathrm{GL}(6, \mathbb{R})$ is the Lie subgroup consisting of matrices of the form

$$g = \begin{pmatrix} \det(\mathbf{B}) & 0 & 0 & 0 \\ 0 & \det(\mathbf{A}) & 0 & 0 \\ 0 & 0 & \mathbf{A} & 0 \\ 0 & 0 & 0 & \mathbf{B} \end{pmatrix}, \tag{15}$$

$$\mathbf{A} = (a_{ij}),\ \mathbf{B} = (b_{ij})\ \in \mathrm{GL}(2, \mathbb{R}).$$

Now let $\mathcal{G}$ denote this G-structure on N. Let $\boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^6)$ be the tautological 1-form on $\mathcal{G}$. Let $\mathfrak{g}$ be the Lie algebra of G. Using the conditions in Proposition 4.1 and the reproducing property of $\boldsymbol{\omega}$, one can show that $\boldsymbol{\omega}$ satisfies the following structure equations, recorded from [Cle02] with a slight change of notation:

$$\mathrm{d}\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} = -\begin{pmatrix} \beta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_0 - \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 0 & \beta_3 & \beta_0 - \beta_1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} + \begin{pmatrix} A_1(\omega^3 - C_1\omega^1) \wedge (\omega^4 - C_2\omega^1) + \omega^5 \wedge \omega^6 \\ \omega^3 \wedge \omega^4 + A_4(\omega^5 - C_3\omega^2) \wedge (\omega^6 - C_4\omega^2) \\ B_1\omega^1 \wedge \omega^2 + C_1\omega^5 \wedge \omega^6 \\ B_2\omega^1 \wedge \omega^2 + C_2\omega^5 \wedge \omega^6 \\ B_3\omega^1 \wedge \omega^2 + C_3\omega^3 \wedge \omega^4 \\ B_4\omega^1 \wedge \omega^2 + C_4\omega^3 \wedge \omega^4 \end{pmatrix}, \tag{16}$$

where the matrix in α and β is a $\mathfrak{g}$-valued 1-form, called a *pseudo-connection* of $\mathcal{G}$; the second term on the right-hand-side is called the *intrinsic torsion* of $\mathcal{G}$.

It is easy to see that the intrinsic torsion above, as a map defined on $\mathcal{G}$, takes values in a 10-dimensional representation of G and is G-equivariant. It is proved in [Cle02] that this representation decomposes into 6 irreducible components, as shown by the following equations, where $u \in \mathcal{G}$ is represented as a column, g is as in (15), and $u \cdot g := g^{-1}u$:

$$
\begin{aligned}
A_1(u\cdot g) = \frac{\det(\mathbf{A})}{\det(\mathbf{B})}A_1(u), \quad & A_4(u\cdot g) = \frac{\det(\mathbf{B})}{\det(\mathbf{A})}A_4(u),\\
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u\cdot g) &= \det(\mathbf{AB})\mathbf{A}^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u),\\
\begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u\cdot g) &= \det(\mathbf{AB})\mathbf{B}^{-1}\begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u),\\
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(u\cdot g) &= \quad \det(\mathbf{B})\mathbf{A}^{-1}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(u),\\
\begin{pmatrix} C_3 \\ C_4 \end{pmatrix}(u\cdot g) &= \quad \det(\mathbf{A})\mathbf{B}^{-1}\begin{pmatrix} C_3 \\ C_4 \end{pmatrix}(u).
\end{aligned}
\tag{17}
$$

Definition 4.2. Let G and $\mathcal{G}$ be as above. The Bäcklund transformation[5] corresponding to $\mathcal{G}$ is said to be *generic* if, at each point $u \in \mathcal{G}$, the intrinsic torsion takes values in a G-orbit with the largest possible dimension.

5. An Estimate of Generality

In this section, we address the problem of generality for *generic* rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. The main ingredients are a *G-structure reduction procedure* described in [Gar89] and a *theorem of Cartan* described in [Bry14].

Lemma 5.1. *Let $(N, \mathcal{B})$ be a nontrivial rank-*1 *Bäcklund transformation relating two hyperbolic Monge-Ampère systems. Let $\mathcal{G}$ be the associated G-structure. If $(N, \mathcal{B})$ is generic, then, at each point $u \in \mathcal{G}$, the intrinsic torsion takes values in an* 8*-dimensional G-orbit.*

Proof. Let

$$W_1 := \mathrm{span}((B_1, B_2), (C_1, C_2)), \quad W_2 := \mathrm{span}((B_3, B_4), (C_3, C_4))$$

at each point $u \in \mathcal{G}$. By (17), the function A_1A_4 and the dimensions of W_1 and W_2 are all invariant under the G-action. Let T denote the intrinsic torsion of $\mathcal{G}$. We claim that, for each $u \in \mathcal{G}$, the G-orbit of $T(u)$ is at most 8-dimensional and that this occurs precisely when W_1 and W_2 are both 2-dimensional. To see why this is true, first note that if one of W_i $(i = 1, 2)$ has dimension less than 2 at $u \in \mathcal{G}$, then the dimension of $u \cdot G$ is at most 7-dimensional. If both W_1, W_2 have dimension 2 at $u \in \mathcal{G}$, then it is easy to show that there exists a unique $g \in G$ such that, at $u' = u \cdot g$,

$$\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} B_3 & C_3 \\ B_4 & C_4 \end{pmatrix} = \begin{pmatrix} \epsilon_2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}$$

where $\epsilon_i = \pm 1$ $(i = 1, 2)$. This completes the proof. □

[5] To be precise, this is a Bäcklund transformation with an ordered pair of characteristic systems.

Let $(N, \mathcal{B})$ be a generic rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. By Lemma 5.1, each point $p \in N$ has a connected open neighborhood $U \subset N$ on which a canonical coframing $(\omega^1, \omega^2, \ldots, \omega^6)$ can be determined. Such a coframing satisfies the equation (16), where all differential forms are defined on U instead of $\mathcal{G}$, and the equations (18), where the sign of each ϵ_i is determined. This motivates the following definition.

Definition 5.1. Let N be a 6-manifold. A coframing $(\omega^1, \omega^2, \ldots, \omega^6)$ defined on an open subset $U \subset N$ is said to be 1-*adapted* (to a generic rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems) if there exist 1-forms α_i, β_i $(i = 0, \ldots, 3)$ and functions A_1, A_4, B_i, C_i $(i = 1, \ldots, 4)$ defined on U such that the equations (16) and (18) are satisfied.

Now we prove the following main theorem that estimates the generality of generic rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Theorem 5.2. *Let N be a 6-manifold. For each $p \in N$, a 1-adapted coframing (Definition 5.1) defined on a small open neighborhood $U \subset N$ of p can be uniquely determined, up to diffeomorphism, by specifying at most 6 functions of 3 variables.*

Proof. Let $U \subset N^6$ be a sufficiently small connected open subset. Suppose that $\boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^6)$ is a 1-adapted coframing on U in the sense of Definition 5.1. It follows that there exist functions P_{ij} $(i = 0, \ldots, 7;\ j = 1, \ldots, 6)$ defined on U such that $\boldsymbol{\omega}$ satisfies (16) and (18) with

$$\alpha_i = P_{ij}\omega^j, \quad \beta_i = P_{i+4,j}\omega^j \qquad (i = 0, \ldots, 3;\ j = 1, \ldots, 6).$$

There is a standard method to determine the generality of such a coframing $\boldsymbol{\omega}$ up to diffeomorphism (see [Bry14]). Our application of such a method involves mainly three steps.

Step 1. By applying $\mathrm{d}^2 = 0$ to (16), we find that P_{ij} are related among themselves and with the coefficients of their exterior derivatives. Repeating this, at a stage, no new relations among the P_{ij} arise.

More explicitly, we can choose s expressions a^α $(\alpha = 1, \ldots, s)$ from P_{ij}, find r expressions b^ρ $(\rho = 1, \ldots, r)$, real analytic functions $F_i^\alpha : \mathbb{R}^{r+s} \to \mathbb{R}$ and $C^i_{jk} : \mathbb{R}^r \to \mathbb{R}$ satisfying $C^i_{jk} + C^i_{kj} = 0$, such that

(A) the equation (16), in general, takes the form

$$\mathrm{d}\omega^i = -\frac{1}{2}C^i_{jk}(a)\omega^j \wedge \omega^k; \tag{19}$$

(B) $\mathrm{d}a^\alpha$, in general, takes the form:

$$\mathrm{d}a^\alpha = F_i^\alpha(a, b)\omega^i; \tag{20}$$

moreover, applying $\mathrm{d}^2 = 0$ to (19) yields identities when we take into account both (19) and (20);

(C) there exist functions $G^\rho_j : \mathbb{R}^{r+s} \to \mathbb{R}$ such that applying $\mathrm{d}^2 = 0$ to (20) yields identities when we replace $\mathrm{d}b^\rho$ by $G^\rho_j \omega^j$ and take into account (19) and (20).

Step 2. For the *tableau of free derivatives* associated to (F^α_i), which is a subspace of $\mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^s)$ defined at each point of $\mathbb{R}^{r+s}$, compute its *Cartan characters* (an array of 6 integers $(s_1, s_2, \ldots, s_6)$) and the dimension δ of its *first prolongation*. For details, see [Bry14]. Moreover, in our case, we verify that $\sum_{i=1}^6 s_i = r$.

Step 3. Restricting to a domain $V \subset \mathbb{R}^{r+s}$ where the Cartan characters are constants, compare $s := \sum_{j=1}^6 j s_j$ with δ. By *Cartan's inequality*, there are two possibilities: either $s = \delta$ (called the *involutive case*) or $s > \delta$.

In the involutive case, one can conclude that (see Theorem 3 in [Bry14]):

> *For any $(a_0, b_0) \in \mathbb{R}^{s+r}$ there exists a coframing $\boldsymbol{\omega}$ and functions $a = (a^\alpha), b = (b^\rho)$ defined on an open neighborhood of $\mathbf{0} \in \mathbb{R}^6$ that satisfy* (19), (20) *and* $(a(\mathbf{0}), b(\mathbf{0})) = (a_0, b_0)$. *Moreover, locally, such a coframing can be uniquely determined up to diffeomorphism by specifying s_k functions of k variables, where s_k is the last nonzero Cartan character.*

In the non-involutive case, which is the case we encounter, a natural step to take is to *prolong* (see [Bry14]) the system by introducing the derivatives of b^ρ, carry out similar steps as the above, and obtain new tableaux of free derivatives with Cartan characters $(\sigma_1, \sigma_2, \ldots, \sigma_6)$.

In practice, however, we do not actually prolong, for it is easy to show that, if s_k is the last nonzero character in $(s_1, \ldots, s_6)$, then $\sigma_j = 0$ $(j > k)$ and $\sigma_k \leqslant s_k$. Using this and the *Cartan-Kuranishi Theorem* ([BCG$^+$13]), one can already conclude that the 'generality' of 1-adapted coframings is *bounded from above* by s_k functions of k variables. (In our case, $k = 3$ and $s_k = 6$.)

For the details of carrying out the steps above, see Appendix A. Most of our calculations are performed using Maple$^{\mathrm{TM}}$. □

Remark 3. Clearly, two 1-adapted coframings that are equivalent under a diffeomorphism correspond to equivalent Bäcklund transformations. Because of this, the upper bound for the 'generality' of 1-adapted coframings in Theorem 5.2 applies to the 'generality' of generic rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Corollary 5.3. *There exist hyperbolic Monge-Ampère systems that are not related to any hyperbolic Monge-Ampère system by a generic rank-*1 *Bäcklund transformation.*

Proof. A hyperbolic Monge-Ampère system, up to contact equivalence, can be uniquely determined by specifying 3 functions of 5 variables.[6]

Note that a generic rank-1 Bäcklund transformation in consideration completely determines the two underlying hyperbolic Monge-Ampère systems (up to equivalence). The conclusion follows. □

6. Classifications and Examples in Higher Cohomogeneity

Following the discussion in the previous section, let $U \subset N^6$ be a sufficiently small connected open subset. Let $\boldsymbol{\omega}$ be a 1-adapted coframing defined on U in the sense of Definition 5.1. One can ask, *when we specify several structure invariants, can we classify the corresponding Bäcklund transformations, if any?*

In the rest of this section, we consider the case when $\epsilon_1 = \epsilon_2 = 1$ in (18), and when A_1 and A_4 (or P_{81} and P_{84} in the new notation) in (16) are specified to be $A_1 = 1$ and $A_4 = -1$.

The following procedure is similar to that in Appendix A. All calculations below are performed using Maple$^{\text{TM}}$.

First, all coefficients in (16) are expressed in terms of the remaining 40 P_{ij}. Defining their derivatives P_{ijk} by

$$\mathrm{d}P_{ij} = P_{ijk}\omega^k$$

and applying the identity $\mathrm{d}^2 = 0$ to (16), we obtain a system of 106 polynomial equations in P_{ij} and P_{ijk}, which implies that

$$\begin{array}{lllll} P_{01} = P_{41}, & P_{02} = P_{42}, & P_{04} = P_{44}, & P_{06} = P_{46}, & P_{11} = 0, \\ P_{12} = 0, & P_{16} = 2P_{46}, & P_{21} = -1, & P_{22} = -1, & P_{23} = 0, \\ P_{35} = 0, & P_{36} = -1, & P_{51} = 0, & P_{52} = 0, & P_{54} = 2P_{44}, \\ P_{61} = 1, & P_{62} = -1, & P_{65} = 0, & P_{73} = 0, & P_{74} = -1. \end{array}$$

Using these relations and repeating the steps above, we obtain a system of 88 equations, which implies that

$$\begin{aligned} P_{25} &= -2(P_{41} + P_{44} - P_{42}), \\ P_{26} &= P_{64}, \\ P_{63} &= -2(P_{41} + P_{42} + P_{46}). \end{aligned}$$

Using these and repeating, we obtain a system of 86 equations for the 17 P_{ij} remaining and 80 of their 102 derivatives. This system implies that

$$P_{31} = P_{32}, \quad P_{41} = -P_{44} - P_{46}, \quad P_{42} = P_{44} - P_{46}, \quad P_{71} = -P_{72}.$$

[6]One can also apply the same method used in the proof of Theorem 5.2 to verify the stronger statement: Locally, a hyperbolic Euler-Lagrange system, which is Monge-Ampère, can be determined uniquely by specifying 1 function of 5 variables. (This is not surprising, as, in our case, a Lagrangian is a function depending on 5 variables.)

Using these and repeating, we obtain a system of 85 equations for the 13 P_{ij} remaining and 64 of their derivatives. This system implies that

$$\begin{aligned}&P_{03} = -1, \quad P_{05} = 0, \quad P_{15} = 0, \quad P_{32} = P_{72}, \quad P_{34} = -1,\\ &P_{43} = 0, \qquad P_{45} = 1, \quad P_{53} = 0, \quad P_{76} = 1.\end{aligned}$$

Using these and repeating, we obtain a system of 61 equations for P_{72}, P_{44}, P_{46}, P_{64} and 22 of their derivatives. Solving this system leads to the two cases below.

Case 1: $P_{72} \neq 0$. In this case, we have

$$P_{44} = 0, \quad P_{46} = 0.$$

Using these and applying $\mathrm{d}^2 = 0$ to the structure equations, we find that

$$P_{64} = \frac{1}{P_{72}}.$$

Using this and repeating, we find that

$$\mathrm{d}(P_{72}) = 0.$$

It follows that the only primary invariant remaining, P_{72}, is a nonzero constant. The structure equations read

$$\begin{aligned}
\mathrm{d}\omega^1 &= \omega^1 \wedge (\omega^3 + \omega^5) + \omega^3 \wedge \omega^4 + \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^2 &= -\omega^2 \wedge (\omega^3 + \omega^5) + \omega^3 \wedge \omega^4 - \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^3 &= \left(\omega^1 + \omega^2 - \frac{1}{P_{72}}\omega^6\right) \wedge \omega^4 + \omega^1 \wedge \omega^2,\\
\mathrm{d}\omega^4 &= -(P_{72}\omega^1 + P_{72}\omega^2 - \omega^6) \wedge \omega^3 + \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^5 &= -\left(\omega^1 - \omega^2 + \frac{1}{P_{72}}\omega^4\right) \wedge \omega^6 + \omega^1 \wedge \omega^2,\\
\mathrm{d}\omega^6 &= (P_{72}\omega^1 - P_{72}\omega^2 + \omega^4) \wedge \omega^5 + \omega^3 \wedge \omega^4.
\end{aligned}$$

This, after the transformation

$$\begin{aligned}&(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6) \mapsto\\ &\qquad (\sqrt{|P_{72}|}\omega^1, \sqrt{|P_{72}|}\omega^2, \omega^3, \sqrt{|P_{72}|}\omega^4, \omega^5, \sqrt{|P_{72}|}\omega^6),\end{aligned}$$

can be readily seen to belong to Case 3D in Clelland's classification (see [Cle02]). According to [Cle02], if $P_{72} < 0$, then $(N, \mathcal{B})$ is a homogeneous Bäcklund transformation relating time-like surfaces of the constant mean curvature

$$H = -\frac{P_{72}}{\sqrt{(P_{72})^2 + 1}}$$

in $\mathbb{H}^{2,1}$; if $P_{72} > 0$, then $(N, \mathcal{B})$ is a homogeneous Bäcklund transformation relating certain surfaces in a 5-dimensional quotient space of the Lie group $\mathrm{SO}^*(4)$.

Case 2: $P_{72} = 0$**.** In this case, all coefficients in (16) are expressed in terms of P_{44}, P_{46} and P_{64}. Applying $\mathrm{d}^2 = 0$ to the structure equations, no new relations between P_{44}, P_{46} and P_{64} arise. Furthermore, 16 of the 18 derivatives of P_{44}, P_{46} and P_{64} are expressed in terms of these three invariants; two derivatives, P_{644} and P_{646}, are free.

It is easy to check that Theorem 3 in [Bry14] applies to $\boldsymbol{\omega}$, the expressions $a = (P_{44}, P_{46}, P_{64})$, $b = (P_{644}, P_{646})$, and the functions C^i_{jk} and F^α_i determined during the calculation above. The corresponding *tableaux of free derivatives* is *involutive* with Cartan characters $(1, 1, 0, 0, 0, 0)$. We have thus proved the following theorem.

Theorem 6.1. *Locally, a generic rank-*1 *Bäcklund transformation* $(N, \mathcal{B})$ *relating two hyperbolic Monge-Ampère systems with its* 1*-adapted coframing satisfying* $\epsilon_1 = \epsilon_2 = 1$ *and* $A_1 = -A_4 = 1$ *can be uniquely determined by specifying* 1 *function of* 2 *variables.*

We can study **Case 2** in greater detail. For convenience, we introduce the following new notation:

$$
\begin{gathered}
R := P_{44} + P_{46}, \quad S := P_{44} - P_{46}, \quad T := P_{64}; \\
T_4 := P_{644}, \quad T_6 := P_{646}.
\end{gathered} \tag{21}
$$

In this new notation, $\boldsymbol{\omega}$ satisfies (16) and (18) where

$$
\begin{aligned}
\alpha_0 &= -R\omega^1 + S\omega^2 - \omega^3 + \tfrac{1}{2}(R+S)\omega^4 + \tfrac{1}{2}(R-S)\omega^6, \\
\beta_0 &= -R\omega^1 + S\omega^2 + \tfrac{1}{2}(R+S)\omega^4 + \omega^5 + \tfrac{1}{2}(R-S)\omega^6, \\
\alpha_1 &= (R-S)\omega^6, \\
\alpha_2 &= (R+S)\omega^5 + T\omega^6 - \omega^1 - \omega^2, \\
\alpha_3 &= -\omega^4 - \omega^6, \\
\beta_1 &= (R+S)\omega^4, \\
\beta_2 &= (R-S)\omega^3 + T\omega^4 + \omega^1 - \omega^2, \\
\beta_3 &= -\omega^4 + \omega^6,
\end{aligned} \tag{22}
$$

and $\epsilon_1 = \epsilon_2 = 1$, $A_1 = -A_4 = 1$.

Moreover, the exterior derivatives of R, S and T are

$$
\begin{aligned}
\mathrm{d}R &= -R^2\omega^1 + (RS-1)\omega^2 + R\omega^3 + \tfrac{1}{2}(R^2 + RS - 1)\omega^4 \\
&\quad + R\omega^5 + \tfrac{1}{2}(R^2 - RS + 1)\omega^6, \\
\mathrm{d}S &= -(RS-1)\omega^1 + S^2\omega^2 - S\omega^3 + \tfrac{1}{2}(S^2 + SR - 1)\omega^4 \\
&\quad - S\omega^5 + \tfrac{1}{2}(SR - S^2 - 1)\omega^6, \\
\mathrm{d}T &= 2(-RT + S)\omega^1 + 2(ST - R)\omega^2 + 2(R^2 - S^2)(\omega^3 + \omega^5) \\
&\quad + T_4\omega^4 + T_6\omega^6.
\end{aligned} \tag{23}
$$

Using these equations, we can study the symmetry of the corresponding Bäcklund transformation $(N, \mathcal{B})$. Let $U \subset N$ be the domain of a 1-adapted coframing $\boldsymbol{\omega}$. Let the map $\Phi : U \to \mathbb{R}^3$ be defined by

$$\Phi(p) = (R(p), S(p), T(p)).$$

Lemma 6.2. *The map Φ can never have rank 0. Moreover, it has*

- *rank 1 if and only if $2RS = 1$ and $T = R^2 + S^2$;*
- *rank 2 if and only if it does not have rank 1 and satisfies either*
  (1) *$2RS = 1$ or*
  (2) *$T_4 = (R + S)(T - 1)$ and $T_6 = (R - S)(T + 1)$;*
- *rank 3 if and only if it does not have rank 1 or 2.*

Proof. In $\mathrm{d}R \wedge \mathrm{d}S \wedge \mathrm{d}T$, the coefficients of $\omega^i \wedge \omega^j \wedge \omega^k$ are polynomials in R, S, T, T_4 and T_6. These coefficients have the common factor $2RS - 1$. Calculating with Maple$^{\mathrm{TM}}$, we find that the coefficients of $\omega^i \wedge \omega^j \wedge \omega^k$ in $(2RS - 1)^{-1}\mathrm{d}R \wedge \mathrm{d}S \wedge \mathrm{d}T$ all vanish if and only if $T_4 = (R + S)(T - 1)$ and $T_6 = (R - S)(T + 1)$. This justifies the conditions for having rank 2. The condition for 'rank-1' can be obtained by setting the coefficients of $\omega^i \wedge \omega^j$ in $\mathrm{d}R \wedge \mathrm{d}S$, $\mathrm{d}R \wedge \mathrm{d}T$ and $\mathrm{d}S \wedge \mathrm{d}T$ to be all zero. By (23), it is clear that $\mathrm{d}R \neq 0$ everywhere; hence, Φ cannot have rank 0. □

Definition 6.1. In the current case, if the corresponding Bäcklund transformation $(U, \mathcal{B})$ has a symmetry whose orbits are of dimension $6 - k$, then it is said to have *cohomogeneity k*.

Case of cohomogeneity-1. This occurs precisely when $\mathrm{rank}(\Phi) = 1$. In this case, locally Φ is a submersion to either branch of the curve in $\mathbb{R}^3$ defined by $2RS = 1$ and $T = R^2 + S^2$. Expressing S and T in terms of R, we have, on $U \subset N$,

$$\begin{aligned} \mathrm{d}R = &-R^2\omega^1 - \frac{1}{2}\omega^2 + R(\omega^3 + \omega^5) \\ &+ \frac{1}{4}(2R^2 - 1)\omega^4 + \frac{1}{4}(2R^2 + 1)\omega^6. \end{aligned} \tag{24}$$

It is clear that $\mathrm{d}R$ is nowhere vanishing. Since R is the only invariant, each constant value of R determines a 5-dimensional submanifold $N_R \subset N$, which has a Lie group structure. The Lie group structure can be determined by setting the right-hand-side of the equation (24) to be zero, obtaining, say,

$$\omega^1 = -\frac{1}{2R^2}\omega^2 + \frac{1}{R}\omega^3 + \frac{2R^2 - 1}{4R^2}\omega^4 + \frac{1}{R}\omega^5 + \frac{2R^2 + 1}{4R^2}\omega^6,$$

then substituting this into the structure equations, yielding equations of $\mathrm{d}\omega^i$ $(i = 2, \ldots, 6)$, expressed in terms of $\omega^2, \ldots, \omega^6$ alone. These are the structure equations on each N_R. Let $X_1, X_2, \ldots, X_5$ be the vector fields tangent to N_R and dual to $\omega^2, \ldots, \omega^6$, such that $\omega^i(X_j) = \delta^i_{j+1}$ $(i - 1, j = 1, 2, \ldots, 5)$.

We obtain the Lie bracket relations:

$$
\begin{aligned}
[X_1, X_2] &= 2X_1 + \frac{1}{R}(X_2 + X_4), \\
[X_1, X_3] &= -\frac{1}{2R}X_1 - \frac{(2R^2-1)}{4R^2}(X_2 - X_4) + \frac{1}{R}X_3, \\
[X_1, X_4] &= 2X_1 + \frac{1}{R}(X_2 + X_4), \\
[X_1, X_5] &= \frac{1}{2R}X_1 + \frac{2R^2+1}{4R^2}(X_2 - X_4) + \frac{1}{R}X_5, \\
[X_2, X_3] &= -X_1 - \frac{1}{R}X_2 - X_3 - X_5, \\
[X_2, X_4] &= 0, \\
[X_2, X_5] &= -\frac{2R^2-1}{2R}X_2 + X_3 + \frac{2R^2+1}{2R}X_4 - X_5, \\
[X_3, X_4] &= -\frac{2R^2-1}{2R}X_2 + X_3 + \frac{2R^2+1}{2R}X_4 - X_5, \\
[X_3, X_5] &= \left(-R^2 + \frac{1}{2}\right)X_2 + RX_3 + \left(R^2 + \frac{1}{2}\right)X_4 - RX_5, \\
[X_4, X_5] &= X_1 - X_3 + \frac{1}{R}X_4 - X_5.
\end{aligned}
$$

Using these relations, it can be verified that X_i $(i = 1, \ldots, 5)$ generate a 5-dimensional Lie algebra that is solvable but not nilpotent. The derived series has dimensions $(5, 3, 1, 0, \ldots)$. In fact, after introducing the following new basis:

$$
\begin{aligned}
\mathbf{e}_1 &= RX_1, \\
\mathbf{e}_2 &= -\frac{1}{2}(X_2 - X_4), \\
\mathbf{e}_3 &= -\frac{1}{2R}X_1 - \frac{2R^2-1}{4R^2}(X_2 - X_4) + \frac{1}{R}X_3, \\
\mathbf{e}_4 &= \frac{1}{2R}X_1 + \frac{2R^2+1}{4R^2}(X_2 - X_4) + \frac{1}{R}X_5, \\
\mathbf{e}_5 &= \frac{1}{R}X_1 + \frac{1}{2R^2}(X_2 + X_4),
\end{aligned}
$$

we obtain the Lie bracket relations:

$$
\begin{aligned}
&[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_3, \quad [\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_4, \quad [\mathbf{e}_1, \mathbf{e}_5] = 2\mathbf{e}_5, \\
&[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_4, \quad [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_3, \quad [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_5,
\end{aligned}
$$

with all $[\mathbf{e}_i, \mathbf{e}_j]$ that are not on this list being zero. An equivalent way of writing these relations is:

$$\begin{aligned}
[\mathbf{e}_1 + i\mathbf{e}_2,\, \mathbf{e}_1 - i\mathbf{e}_2] &= 0,\\
[\mathbf{e}_1 + i\mathbf{e}_2,\, \mathbf{e}_3 + i\mathbf{e}_4] &= 2(\mathbf{e}_3 + i\mathbf{e}_4),\\
[\mathbf{e}_1 - i\mathbf{e}_2,\, \mathbf{e}_3 + i\mathbf{e}_4] &= 0,\\
[\mathbf{e}_1 + i\mathbf{e}_2,\, \mathbf{e}_5] &= 2\mathbf{e}_5,\\
[\mathbf{e}_3 + i\mathbf{e}_4,\, \mathbf{e}_3 - i\mathbf{e}_4] &= -2i\mathbf{e}_5,\\
[\mathbf{e}_3 + i\mathbf{e}_4,\, \mathbf{e}_5] &= 0.
\end{aligned}$$

Now, it is easy to see that the Lie algebra $\bigoplus_{i=1}^{5} \mathbb{R}\mathbf{e}_i$ is isomorphic to the Lie algebra generated by the real and imaginary parts of the vector fields

$$\partial_w, \quad e^{2w}(\partial_z + i\bar{z}\partial_\lambda), \quad e^{2(w+\bar{w})}\partial_\lambda$$

on $\mathbb{R} \times \mathbb{C}^2$ with coordinates $(\lambda; z, w)$. In fact, an isomorphism is induced by the correspondence

$$\mathbf{e}_1 + i\mathbf{e}_2 \mapsto \partial_w, \quad \mathbf{e}_3 + i\mathbf{e}_4 \mapsto e^{2w}(\partial_z + i\bar{z}\partial_\lambda), \quad \mathbf{e}_5 \mapsto e^{2(w+\bar{w})}\partial_\lambda.$$

Next, we describe the hyperbolic Monge-Ampère systems related by the Bäcklund transformation being considered.

Proposition 6.1. *A Bäcklund transformation in the current (cohomogeneity-*1*) case is an auto-Bäcklund transformation of a homogeneous Euler-Lagrange system.*

Proof. This proof is in two parts. First, we show that the underlying two hyperbolic Monge-Ampère systems are equivalent and are homogeneous. Second, by computing their local invariants, we verify that they are hyperbolic *Euler-Lagrange systems* in the sense of [BGG03].

Using the structure equations on $U \subset N$, if we let $(\theta^0, \theta^1, \ldots, \theta^4)$ be either

$$\big(S\omega^1, -R(\omega^1 - \omega^4) + \omega^3, S\omega^4, -R(\omega^1 - \omega^6) + \omega^5, S\omega^6\big) \tag{25}$$

or

$$\big(-R\omega^2, S(\omega^2 + \omega^4) - \omega^3, R\omega^4, S(\omega^2 - \omega^6) - \omega^5, -R\omega^6\big)\,, \tag{26}$$

then we can verify (with Maple$^{\mathrm{TM}}$) that θ^i $(i = 0, \ldots, 4)$, in both cases, satisfy the same structure equations:

$$\begin{aligned}
\mathrm{d}\theta^0 &= -2(\theta^1 + \theta^3) \wedge \theta^0 + \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4,\\
\mathrm{d}\theta^1 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3,\\
\mathrm{d}\theta^2 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4,\\
\mathrm{d}\theta^3 &= \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^3,\\
\mathrm{d}\theta^4 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4.
\end{aligned} \tag{27}$$

It is easy to verify that (27) are the structure equations on a 5-manifold M with a hyperbolic Monge-Ampère ideal $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$. It follows that the expressions (25) and (26) correspond to the pull-back of θ^i under two distinct submersions $\pi_1, \pi_2 : U \to M$. It is easy to see that $(U, \mathcal{B}; \pi_1, \pi_2)$ is an auto-Bäcklund transformation of the system $(M, \mathcal{I})$. Moreover, $(M, \mathcal{I})$ is *homogeneous*, since all coefficients in (27) are constants.

Next, we verify that $(M, \mathcal{I})$ is a *hyperbolic Euler-Lagrange system*. In [BGG03], it is proved that a hyperbolic Monge-Ampère system is Euler-Lagrange if and only if the invariant tensor S_2 vanishes (see Proposition 3.2). To compute S_2 in the current case, we choose a new coframing $\boldsymbol{\eta} = (\eta^i)$ and 1-forms (ϕ_α) below:

$$\begin{aligned}(\eta^0, \eta^1, \eta^2, \eta^3, \eta^4) = (\sqrt{2}\theta^0,\ \sqrt{2}\theta^1,\ \theta^1 + \theta^2 - \theta^0,\\ \theta^3 + \theta^4 - \theta^0,\ \sqrt{2}(\theta^4 - \theta^0)),\end{aligned}$$

$$\begin{aligned}
&\phi_0 = \frac{1}{\sqrt{2}}\eta^1 + \eta^3 - \frac{1}{\sqrt{2}}\eta^4, && \\
&\phi_1 = -\sqrt{2}\eta^0 - \eta^3 - \frac{1}{\sqrt{2}}\eta^4, && \phi_2 = \eta^0 + \sqrt{2}\eta^3, \\
&\phi_3 = -2\eta^0 + \eta^1 - \frac{1}{\sqrt{2}}\eta^2 - \sqrt{2}\eta^3 - \eta^4, && \phi_4 = \phi_0 - \phi_1, \\
&\phi_5 = -\eta^3 - \frac{1}{\sqrt{2}}\eta^4, && \phi_6 = \frac{1}{\sqrt{2}}\eta^3, \\
&\phi_7 = -\eta^0 + \eta^1 - \sqrt{2}\eta^2 - \sqrt{2}\eta^3, && \phi_8 = \phi_0 - \phi_5.
\end{aligned} \tag{28}$$

These η^i $(i = 0, 1, \ldots, 4)$ and ϕ_α $(\alpha = 1, \ldots, 8)$ are chosen such that they satisfy the structure equations (5), and such that S_1 and S_2 are as simple as possible. One can verify that, under this choice,

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \mathbf{0}.$$

This completes the proof. □

Remark 4. The Monge-Ampère system $(M, \mathcal{I})$ considered in Proposition 6.1 cannot, by a contact transformation, be put in the form

$$z_{xy} = f(x, y, z, z_x, z_y).$$

This is because $\mathrm{rank}(S_1) = 2$ and $S_2 = \mathbf{0}$, which implies that neither of the two characteristic systems of $(M, \mathcal{I})$ contains a rank-1 integrable subsystem.

Now, one may wonder whether the homogeneous Monge-Ampère system $(M, \mathcal{I})$ considered in Proposition 6.1 has a symmetry of dimension greater than 5. Using the *method of equivalence*, we prove that it is not the case.

Proposition 6.2. *The hyperbolic Euler-Lagrange system in Proposition 6.1 has a symmetry of dimension* 5*. In addition, any such symmetry is induced from a symmetry of the Bäcklund transformation* $(N, \mathcal{B})$*.*

Proof. Let $(M, \mathcal{I})$ denote the Euler-Lagrange system being considered. To show that $(M, \mathcal{I})$ has a 5-dimensional symmetry, it suffices to show that there is a canonical way to determine a local coframing on M. This can be achieved by applying the *method of equivalence*. For details, see Appendix B.

By (25) and (26), it is easy to see that the fibers of $\pi_i : N \to M$ $(i = 1, 2)$ are everywhere transversal to the level sets of the functions R. The second half of the statement follows. □

To end the discussion of the cohomogeneity-1 case, we integrate the structure equations (27) to express the corresponding hyperbolic Euler-Lagrange system in local coordinates.

Proposition 6.3. *The hyperbolic Monge-Ampère system* $(M, \mathcal{I})$ *with the differential ideal* $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$*, where* θ^i *satisfy* (27)*, is equivalent to the following hyperbolic Monge-Ampère PDE up to a contact transformation:*

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} = 0, \tag{29}$$

where $A = 2z_x + y$ *and* $B = 2z_y - x$*.*

Proof. Let $U \subset M$ be a domain on which θ^i are defined. One can verify, using the structure equations (27), that the 1-forms $\theta^1 - \theta^2 - \theta^3 - \theta^4$ and $\theta^1 + \theta^3$ are closed. Hence, by shrinking U if needed, there exist functions P, Q defined on U such that

$$\mathrm{d}P = -(\theta^1 + \theta^3), \quad \mathrm{d}Q = \theta^1 - \theta^2 - \theta^3 - \theta^4.$$

Moreover, if we let $\Theta = \theta^2 + i\theta^4$, then, by a straightforward calculation, we obtain

$$\mathrm{d}\Theta = \mathrm{d}(P + iQ) \wedge \Theta.$$

It follows that there exist functions X, Y on U such that

$$\Theta = e^{P+iQ}\mathrm{d}(X + iY).$$

Equivalently, we have

$$\theta^2 = e^P(\cos Q\ \mathrm{d}X - \sin Q\ \mathrm{d}Y),$$
$$\theta^4 = e^P(\sin Q\ \mathrm{d}X + \cos Q\ \mathrm{d}Y).$$

Now we can express $\theta^1, \ldots, \theta^4$ completely in terms of the functions X, Y, P, Q. By verifying the equality

$$\mathrm{d}(e^{-2P}\theta^0) = e^{-2P}(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4), \tag{30}$$

we notice that the right-hand-side of (30) is a symplectic form on a 4-manifold on which $\theta^1, \ldots, \theta^4$ are well-defined, by (27). Thus, by the theorem of Darboux, locally there exist functions x, y, p, q such that the right-hand-side of (30) is equal to $\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q$.

In fact, in the $XYPQ$-coordinates, the right-hand-side of (30) is equal to

$$\mathrm{d}\left(\frac{e^{-P}}{2}(\cos Q+\sin Q)+\frac{Y}{2}\right)\wedge\mathrm{d}X \\ +\mathrm{d}\left(\frac{e^{-P}}{2}(\cos Q-\sin Q)-\frac{X}{2}\right)\wedge\mathrm{d}Y.$$

As a result, we can set

$$x=X,\quad p=-\frac{e^{-P}}{2}(\cos Q+\sin Q)-\frac{Y}{2},$$
$$y=Y,\quad q=-\frac{e^{-P}}{2}(\cos Q-\sin Q)+\frac{X}{2},$$

and write

$$e^{-2P}\theta^0=\mathrm{d}z-p\mathrm{d}x-q\mathrm{d}y,$$

for some function z, independent of x,y,p,q. From these expressions, it is clear that $2p+y$ and $2q-x$ cannot simultaneously vanish.

Now let $A=2p+y$ and $B=2q-x$. We can express $\theta^1\wedge\theta^2$ in terms of x,y,z,p,q:

$$\frac{(A^2-B^2)(\mathrm{d}p\wedge\mathrm{d}y-\mathrm{d}x\wedge\mathrm{d}q)+(A+B)^2\mathrm{d}x\wedge\mathrm{d}p+(A-B)^2\mathrm{d}y\wedge\mathrm{d}q}{(A^2+B^2)^2}.$$

Multiplying this expression by $(A^2+B^2)^2$ then subtracting the result by $(A^2+B^2)(\mathrm{d}x\wedge\mathrm{d}p+\mathrm{d}y\wedge\mathrm{d}q)$, we obtain

$$(A^2-B^2)(\mathrm{d}p\wedge\mathrm{d}y-\mathrm{d}x\wedge\mathrm{d}q)+2AB(\mathrm{d}x\wedge\mathrm{d}p+\mathrm{d}q\wedge\mathrm{d}y).$$

The vanishing of this 2-form on integral surfaces (satisfying the independence condition $\mathrm{d}x\wedge\mathrm{d}y\neq 0$) implies that z must satisfy the equation (29). □

Case of cohomogeneity-2. By Lemma 6.2, this case can only occur when $2RS=1$ and $R^2+S^2=T$ do not both hold and either

(1) $2RS=1$ or
(2) $T_4=(R+S)(T-1)$, $T_6=(R-S)(T+1)$

holds. We now focus on the latter case.

Proposition 6.4. *When Φ has rank 2, and when $T_4=(R+S)(T-1)$ and $T_6=(R-S)(T+1)$, the map $\Phi:N\to\mathbb{R}^3$ has its image contained in a surface that is defined by either*

$$\frac{R^2+S^2-T}{2RS-1}$$

or its reciprocal being a constant.

Proof. First note that R^2+S^2-T and $2RS-1$ cannot be both zero, for this would reduce to the cohomogeneity-1 case; hence, the conclusion has meaning. To see that this statement is true, note that, in the current case, the pull-back of $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$ via Φ to N are linearly dependent. To be precise, the 1-form

$$\begin{aligned}\theta = &-2(R^2S - S^3 + ST - R)\mathrm{d}R \\ &+ 2(R^3 - RS^2 - RT + S)\mathrm{d}S + (2RS - 1)\mathrm{d}T\end{aligned}$$

equals to zero when pulled back to N. Since the tangent map Φ_* has rank 2, this can only occur when $\theta \wedge \mathrm{d}\theta = 0$. It follows that θ is integrable. In fact, it is easy to verify that the primitives of $(2RS-1)^{-2}\theta$ and $(R^2+S^2-T)^{-2}\theta$ are, respectively, the function

$$\frac{R^2 + S^2 - T}{2RS - 1}$$

and its reciprocal when $2RS-1$ and R^2+S^2-T are, respectively, nonzero. This completes the proof. □

We now study the symmetry of the Bäcklund transformation $(N, \mathcal{B})$ being considered. Let X_i $(i = 1, 2, \ldots, 6)$ be the vector fields defined on $U \subset N$ that are dual to ω^i $(i = 1, 2, \ldots, 6)$. Using the expressions of $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$, it is easy to see that the rank-4 distribution on U annihilated by $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$ is spanned by the vector fields

$$\begin{aligned}Y_1 &= RX_2 + SX_1 + X_5, & Y_2 &= \frac{1}{2}(X_1 - X_2) + X_4, \\ Y_3 &= X_3 - X_5, & Y_4 &= \frac{1}{2}(X_1 + X_2) + X_6.\end{aligned}$$

These vector fields generate a 4-dimensional Lie algebra $\mathfrak{l}$ with

$$\begin{aligned}[][Y_1, Y_2] &= \frac{R+S}{2}Y_3 + Y_4, \\ [Y_1, Y_3] &= 0, \\ [Y_1, Y_4] &= -Y_2 + \frac{R-S}{2}Y_3, \\ [Y_2, Y_3] &= (R+S)Y_3 + 2Y_4, \\ [Y_2, Y_4] &= Y_1 + (R-S)Y_2 + \left(\frac{1}{2} - T\right)Y_3 - (R+S)Y_4, \\ [Y_3, Y_4] &= 2Y_2 - (R-S)Y_3.\end{aligned}$$

It is easy to verify that $2Y_1 + Y_3$ belongs to the center of $\mathfrak{l}$. The quotient algebra $\mathfrak{q} = \mathfrak{l}/\mathbb{R}(2Y_1+Y_3)$, with the basis $\mathbf{e}_1 = [Y_2]$, $\mathbf{e}_2 = [Y_3]$ and $\mathbf{e}_3 = [Y_4]$, satisfies

$$\begin{aligned}[][\mathbf{e}_1, \mathbf{e}_2] &= (R+S)\mathbf{e}_2 + 2\mathbf{e}_3, \\ [\mathbf{e}_1, \mathbf{e}_3] &= (R-S)\mathbf{e}_1 - T\mathbf{e}_2 - (R+S)\mathbf{e}_3, \\ [\mathbf{e}_2, \mathbf{e}_3] &= 2\mathbf{e}_1 + (S-R)\mathbf{e}_2.\end{aligned}$$

According to the classification of 3-dimensional Lie algebras, see Lecture 2 in [Bry95] for example, to identify the Lie algebra $\mathfrak{q}$, it suffices to find a normal form of the matrix (note that it is symmetric)

$$C = \begin{pmatrix} 2 & S-R & 0 \\ S-R & T & R+S \\ 0 & R+S & 2 \end{pmatrix}$$

under the transformation $C \mapsto \det(A^{-1})ACA^T$, where $A \in \mathrm{GL}(3,\mathbb{R})$. Note that $\det(C) = -2(R^2+S^2-T)$. We have:

Proposition 6.5. *If $R^2+S^2 < T$, then $\mathfrak{q}$ is isomorphic to $\mathfrak{so}(3,\mathbb{R})$. If $R^2+S^2 > T$, then $\mathfrak{q}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. If $R^2+S^2 = T$, then $\mathfrak{q}$ is isomorphic to the solvable Lie algebra with a basis x_1, x_2, x_3 satisfying $[x_2,x_3] = x_1$, $[x_3,x_1] = x_2$ and $[x_1,x_2] = 0$.*

Proof. After a transformation of the form above, C can be put in the form

$$C' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & T-R^2-S^2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

By [Bry95], the conclusion follows immediately. □

Now consider the case when $\mathfrak{q}$ is solvable, that is, when $R^2+S^2 = T$. By the cohomogeneity-2 assumption, we must have $2RS \neq 1$. We proceed to identify the Monge-Ampère systems related by such a Bäcklund transformation.

If we let $(\theta^0, \theta^1, \ldots, \theta^4)$ be

$$(S\omega^1, -R(\omega^1-\omega^4)+\omega^3, S\omega^4, -R(\omega^1-\omega^6)+\omega^5, S\omega^6)$$

and let F be defined by

$$F = \frac{2RS-1}{2S^2},$$

for which to have meaning we need to restrict to a domain on which $S \neq 0$, then the structure equations on N would imply

$$\begin{aligned}
\mathrm{d}\theta^0 &= \theta^0 \wedge (2\theta^1 - F\theta^2 + 2\theta^3 - F\theta^4) + \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^1 &= -F\theta^0 \wedge (\theta^2+\theta^4) - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + (F+1)\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= -2F\theta^0 \wedge \theta^2 - \theta^1 \wedge \theta^2 - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 \\
&\quad + (1-F)\theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= F\theta^0 \wedge (\theta^2+\theta^4) + \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3 + (F-1)\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^4 &= -2F\theta^0 \wedge \theta^4 + \theta^1 \wedge \theta^2 - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 \\
&\quad + (F+1)\theta^2 \wedge \theta^4 - \theta^3 \wedge \theta^4,
\end{aligned} \tag{31}$$

and

$$\mathrm{d}F = 2F^2(2\theta^0 - \theta^2 - \theta^4) + 2F(\theta^1+\theta^3), \quad (F \neq 0). \tag{32}$$

It can be verified that, in the equations (31) and (32), the exterior derivative of the right-hand-sides are zero, by taking into account these equations themselves. By the construction of the θ^i ($i = 0, \dots, 4$), it follows that (31) and (32) are the structure equations of one of the Monge-Ampère systems being related by the Bäcklund transformation $(N, \mathcal{B})$.

On the other hand, it is easy to verify that the transformation

$$(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4; F) \mapsto (-\theta^0, \theta^3, -\theta^4, \theta^1, -\theta^2; -F)$$

leaves (31) and (32) unchanged. Thus, by applying such a transformation, if needed, we can assume that $F > 0$.

Proposition 6.6. *The hyperbolic Monge-Ampère system $(M, \mathcal{I})$ with the differential ideal $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$, where θ^i satisfy* (31) *and* (32), *corresponds to the following hyperbolic Monge-Ampère PDE up to a contact transformation:*

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} + (A^2 + B^2)^2 = 0. \tag{33}$$

where $A = z_x - y$, $B = z_y + x$.

Proof. The proof is similar to that of Proposition 6.3. First it is easy to verify that the 1-forms $F(-2\theta^0 + \theta^2 + \theta^4) - \theta^1 - \theta^3$ and $\theta^1 - \theta^2 - \theta^3 - \theta^4$ are closed. Consequently, locally there exist functions f, g such that

$$\begin{aligned} \mathrm{d}f &= F(-2\theta^0 + \theta^2 + \theta^4) - \theta^1 - \theta^3, \\ \mathrm{d}g &= \theta^1 - \theta^2 - \theta^3 - \theta^4. \end{aligned}$$

Now the expression of $\mathrm{d}F$ can be written as $\mathrm{d}F = -2F\mathrm{d}f$. This implies that there exists a constant $C > 0$ such that $F = Ce^{-2f}$. Using the ambiguity in f (as f is determined up to an additive constant), we can arrange that $C = 1$. In addition, if we let $\Theta = e^{-f}(\theta^2 + i\theta^4)$, it is easy to verify that Θ is integrable. To be explicit,

$$\mathrm{d}\Theta = i\ \mathrm{d}g \wedge \Theta.$$

Thus, there exist functions X, Y such that $\Theta = e^{ig}(\mathrm{d}X + i\mathrm{d}Y)$. From this we obtain

$$\begin{aligned} \theta^2 &= e^f(\cos g\ \mathrm{d}X - \sin g\ \mathrm{d}Y), \\ \theta^4 &= e^f(\sin g\ \mathrm{d}X + \cos g\ \mathrm{d}Y). \end{aligned}$$

Using these, differentiating $\theta^1 + \theta^3$ gives

$$\mathrm{d}(\theta^1 + \theta^3) = 2\ \mathrm{d}X \wedge \mathrm{d}Y.$$

This implies that there exists a function Z, independent of X, Y, f, g, such that

$$\theta^1 + \theta^3 = \mathrm{d}Z + X\mathrm{d}Y - Y\mathrm{d}X.$$

Now, $\theta^0, \theta^1, \dots, \theta^4$ can be completely expressed in terms of the functions X, Y, Z, f, g. In particular,

$$\begin{aligned}-2e^{-2f}\theta^0 = \mathrm{d}(Z+f) &- (Y + e^{-f}(\sin g + \cos g))\mathrm{d}X \\ &+ (X + e^{-f}(\sin g - \cos g))\mathrm{d}Y.\end{aligned}$$

If we make the substitution

$$\begin{aligned} x &= X, & p &= e^{-f}(\cos g + \sin g) + Y, \\ y &= Y, & q &= e^{-f}(\cos g - \sin g) - X, \\ z &= Z + f, \end{aligned}$$

the contact form θ^0 is then a nonzero multiple of $\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$. The 2-form $\theta^3 \wedge \theta^4$, when each $\mathrm{d}z$ is replaced by $p\mathrm{d}x + q\mathrm{d}y$, can be expressed as

$$\begin{aligned}\theta^3 \wedge \theta^4 \equiv \frac{1}{8}e^{4f}\Big\{&(A^2 - B^2)(\mathrm{d}p \wedge \mathrm{d}y - \mathrm{d}x \wedge \mathrm{d}q) \\ &+ (A+B)^2 \ \mathrm{d}q \wedge \mathrm{d}y \\ &- (A-B)^2 \ \mathrm{d}x \wedge \mathrm{d}p \\ &+ (A^2+B^2)^2 \mathrm{d}x \wedge \mathrm{d}y\Big\}\end{aligned} \qquad \mathrm{mod}\ \theta^0,$$

where $A = p - y$, $B = q + x$. Note that, by construction, A, B cannot be simultaneously zero. The equation (33) follows. □

In the current case, there remain several obvious questions to investigate. *What is the Monge-Ampère system corresponding to* $\langle \omega^2, \omega^3 \wedge \omega^4, \omega^5 \wedge \omega^6 \rangle$*? Are the Monge-Ampère systems being Bäcklund-related Euler-Lagrange? Is the Bäcklund transformation an auto-Bäcklund transformation?* Answers to these questions can be obtained in a similar way as in the cohomogeneity-1 case. We thus have them summarized in the following remark, omitting the details of calculation.

Remark 5. **A.** Whenever $R \neq 0$,

$$(R\omega^2, S(\omega^2 + \omega^4) - \omega^3, -R\omega^4, S(\omega^2 - \omega^6) - \omega^5, R\omega^6)$$

form a coframing defined on a 5-manifold. The system

$$\langle \omega^2, \omega^3 \wedge \omega^4, \omega^5 \wedge \omega^6 \rangle$$

descends to correspond to the same equation (33) up to a contact transformation. The system $(N, \mathcal{B})$ is therefore an auto-Bäcklund transformation of the equation (33).

B. One can verify that the hyperbolic Monge-Ampère system in Proposition 6.6 is Euler-Lagrange. In fact, by a transformation of the θ^i ($i = 0, \dots, 4$), the structure equations (31) can be put in the form of

(5) with

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \mathbf{0}.$$

Applying the procedure described in Appendix B, we find the corresponding Monge-Ampère invariants

$$Q_{01}Q_{04} - Q_{02}Q_{03} = -\frac{1}{2}(1+F^2) \neq 0, \quad Q_{00} = 0.$$

Using this and the expression of $\mathrm{d}F$, one can show (in a way that is similar to the proof of Proposition 6.2) that the underlying Euler-Lagrange system has a symmetry of dimension 4. Such a symmetry is induced from the symmetry of the Bäcklund transformation $(N, \mathcal{B})$.

C. Similar to the reason in Remark 4, (33) is not contact equivalent to any hyperbolic Monge-Ampère PDE of the form

$$z_{xy} = f(x, y, z, z_x, z_y).$$

7. Concluding Remarks

7.1. Regarding Classification in the Rank-1 Case.

The method outlined in Section 6 can be carried further, for example, by putting weaker restrictions on the invariants. Which new subclasses of Bäcklund transformations relating two hyperbolic Monge-Ampère systems can be classified?

Equation (29) admits the trivial solution $z(x, y) = 0$. Which solution do we obtain by applying the Bäcklund transformation found in Proposition 6.1 to this trivial solution? Does this give rise to a 1-soliton solution to (29)?

7.2. Regarding Monge-Ampère Invariants.

It is interesting to ask which pairs of hyperbolic Monge-Ampère systems may be related by a rank-1 Bäcklund transformation. This question can be partially answered by studying how obstructions to the existence of Bäcklund transformations may be expressed in terms of the invariants of the underlying hyperbolic Monge-Ampère systems. Some relevant results are presented in [Hu19].

7.3. Bäcklund Transformations of Higher Ranks.

Among the examples discussed in [RS02], a Bäcklund transformation relating solutions of the *hyperbolic Tzitzeica equation* is particularly interesting. The *hyperbolic Tzitzeica equation* is the second-order equation for $h(x, y)$:

$$(\ln h)_{xy} = h - h^{-2}. \tag{34}$$

This equation was discovered by Tzitzeica in his study of *hyperbolic affine spheres* in the affine 3-space $\mathbb{A}^3$ (see [Tzi08] and [Tzi09]). He found that the system in α, β and h,

$$\begin{cases} \alpha_x = (h_x\alpha + \lambda\beta)h^{-1} - \alpha^2, \\ \alpha_y = \beta_x = h - \alpha\beta, \\ \beta_y = (h_y\beta + \lambda^{-1}\alpha)h^{-1} - \beta^2, \end{cases} \tag{35}$$

where λ is an arbitrary nonzero constant, is a Bäcklund transformation relating solutions of (34). More explicitly, if h solves the hyperbolic Tzitzeica equation (34), then, substituting it in the system (35), one obtains a compatible first-order PDE system for α and β, whose solutions can be found by solving ODEs; for each solution (α, β), the function

$$\bar{h} = -h + 2\alpha\beta$$

also satisfies the hyperbolic Tzitzeica equation (34).

Unlike the systems (1) and (2), substituting a solution h of (34) into (35) (with fixed λ) yields a system whose solutions depend on 2 parameters instead of 1. Using our terminology (see Definition 2.10), one can verify that the system (35) corresponds to a *rank*-2 Bäcklund transformation.

Furthermore, in [AF15], it is shown that the following hyperbolic Monge-Ampère equation

$$z_{xy} = \frac{\sqrt{1 - z_x^2}\sqrt{1 - z_y^2}}{\sin z}$$

and the wave equation $z_{xy} = 0$ admit no rank-1 Bäcklund transformation relating their solutions, but a rank-2 Bäcklund transformation does exist.

In a future paper, we will present a partial classification of homogeneous rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. (It will turn out that the rank-2 Bäcklund transformation corresponding to (35) is nonhomogeneous.) Based on our classification so far, we expect that those homogeneous Bäcklund transformations (relating two hyperbolic Monge-Ampère systems) that are 'genuinely' rank-2 are quite few.

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Appendix A. Calculations in Theorem 5.2

This Appendix supplements the proof of Theorem 5.2 by providing more details of calculation. Most calculations below are computed using Maple$^{\text{TM}}$.

First consider the case when, on U, $\epsilon_1 = \epsilon_2 = 1$. Since P_{24}, P_{33}, P_{66}, P_{75} never appear in the equation (16), we can set them all to zero. Since P_{14} and P_{23} only appear in the term $(P_{14} - P_{23})\omega^3 \wedge \omega^4$, we can set $P_{14} = 0$. For similar reasons, we can set $P_{13}, P_{55}, P_{56} = 0$. For convenience, we rename A_1 as P_{81} and A_4 as P_{84}. Now there are 42 functions P_{ij} remaining, and they are determined.

For each P_{ij}, there exist functions P_{ijk} defined on U satisfying

$$\mathrm{d}(P_{ij}) = P_{ijk}\omega^k.$$

We call these P_{ijk} the *derivatives* of P_{ij}.

Now, applying $\mathrm{d}^2 = 0$ to the equation (16), we obtain 106 polynomial equations expressed in terms of all 42 P_{ij} and 186 of all 252 P_{ijk}. These equations imply:

$$\begin{aligned}
&P_{01} = P_{41} - P_{51}, && P_{02} = P_{42} - P_{52}, && P_{03} = P_{53} - P_{43} - P_{81},\\
&P_{04} = P_{54} - P_{44}, && P_{05} = P_{15} - P_{45} - P_{84}, && P_{11} = -P_{51},\\
&P_{12} = -P_{52}, && P_{21} = P_{84}, && P_{22} = -1,\\
&P_{35} = 0, && P_{36} = -1, && P_{61} = 1,\\
&P_{62} = -P_{81}, && P_{73} = 0, && P_{74} = -1.
\end{aligned}$$

With these relations, all coefficients in (16) can be expressed in terms of 27 P_{ij}. Repeating the steps above by defining the derivatives P_{ijk} (now 162 in all) and applying $\mathrm{d}^2 = 0$ to (16), we obtain a system of 91 polynomial equations in these 27 P_{ij} and 124 of all 162 P_{ijk}, which imply

$$\begin{aligned}
P_{31} &= -P_{32}P_{84} - P_{15} - P_{34} - 2P_{43} - 2P_{45} + P_{53} + P_{76} - P_{81} - P_{84},\\
P_{72} &= -P_{71}P_{81} - P_{15} + P_{34} + 2P_{43} + 2P_{45} - 3P_{53} - P_{76} + P_{81} + P_{84}.
\end{aligned}$$

Using these relations and repeating the steps above, we obtain

$$P_{06} = P_{16} - P_{46}.$$

All coefficients in (16) are then expressed in terms of 24 P_{ij}.

Now, corresponding to the remaining 24 P_{ij} are 144 derivatives P_{ijk}. Applying $\mathrm{d}^2 = 0$ to (16) yields a system of 88 polynomial equations, expressed in terms of the 24 P_{ij} and 122 of the 144 derivatives P_{ijk}. This system can be solved for P_{ijk}; in the solution, all P_{ijk} are expressed explicitly in terms of the 24 P_{ij} and 64 P_{ijk} that are 'free'.

Let $a = (a^\alpha)$ $(\alpha = 1, \ldots, 24)$ stand for the 24 remaining P_{ij}; let $b = (b^\rho)$ $(\rho = 1, \ldots, 64)$ stand for the 64 'free' P_{ijk}. We already have

$$\mathrm{d}\omega^i = -\frac{1}{2}C^i_{jk}(a)\omega^j \wedge \omega^k, \tag{36}$$

$$\mathrm{d}a^\alpha = F^\alpha_i(a,b)\omega^i, \tag{37}$$

for some real analytic functions F^α_i and C^i_{jk} satisfying $C^i_{jk} + C^i_{kj} = 0$.

Now compute the exterior derivatives

$$\mathrm{d}(F^\alpha_i(a,b)\omega^i), \quad \alpha = 1, \ldots, 24,$$

and take into account (36) and (37). From this we obtain 2-forms Ω^α that are linear combinations of $\mathrm{d}b^\rho \wedge \omega^i$ and $\omega^i \wedge \omega^j$. Let $\hat{\Omega}^\alpha$ denote the part of

Ω^α consisting of linear combinations of $\mathrm{d}b^\rho \wedge \omega^i$ only. Replacing $\mathrm{d}b^\rho$ in $\hat{\Omega}^\alpha$ by $G^\rho_i\omega^i$ defines a linear map

$$\phi : \mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^{64}) \to \Lambda^2(\mathbb{R}^6)^* \otimes \mathbb{R}^{24}$$

at each point of U.

Let $[\Omega]$ denote the equivalence class of (Ω^α) in the cokernel of ϕ. One can show that $[\Omega]$ must vanish and that its vanishing leads to a system of 35 equations for a and b. This system can be solved for 12 of the 64 components of b. Apply such a solution and update a^α, b^ρ and the functions F^α_i accordingly.

It is not difficult to verify, using Maple™, that the updated a^α ($\alpha = 1, \ldots, 24$), b^ρ ($\rho = 1, \ldots, 52$), C^i_{jk} and F^α_i satisfy the conditions **(A)**-**(C)** in **Step 1**.

For **Steps 2 and 3**, calculation shows that the *tableaux of free derivatives* has Cartan characters

$$(s_1, s_2, s_3, s_4, s_5, s_6) = (24, 22, 6, 0, 0, 0)$$

and the dimension of its first prolongation

$$\delta = 64 < s_1 + 2s_2 + 3s_3 + 4s_4 + 5s_5 + 6s_6 = 86.$$

The cases when, on U, ϵ_1 and ϵ_2 take other values follow similar steps. In each of these cases, the last nonzero Cartan character, computed at a corresponding stage, is $s_3 = 6$.

Appendix B. Invariants of an Euler-Lagrange System

This Appendix supplements the proof of Proposition 6.2.

We start with the G_1-structure $\pi : \mathcal{G}_1 \to M$ of a hyperbolic Monge-Ampère system $(M, \mathcal{I})$ (see [BGG03] or Section 3). Assume that $S_2 = \mathbf{0}$ (i.e., the Euler-Lagrange case).

Recall that the 2×2-matrix $S_1 : \mathcal{G}_1 \to \mathfrak{gl}(2, \mathbb{R})$ is equivariant under the G_1-action. By (9) and (10), it is easy to see that, when $\det(S_1(u)) > 0$ (resp., $\det(S_1(u)) < 0$) at $u \in \mathcal{G}_1$, the same is true for $\det(S_1(u \cdot g))$ for all $g \in G_1$, and the matrix $S_1(u)$ lies in the same G_1-orbit as $\mathrm{diag}(1, 1)$ (resp., $\mathrm{diag}(1, -1)$).

Now assume that $\det(S_1) > 0$ holds on $\pi^{-1}U \subset \mathcal{G}$ for some domain $U \subset M$. By the discussion above, we can reduce to a subbundle $\mathcal{H} \subset \mathcal{G}_1$ defined by $S_1 = \mathrm{diag}(1, 1)$.

It is easy to see that $\mathcal{H}$ is an H-structure on U where

$$H = \left\{ \left(\begin{array}{ccc} \epsilon & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \epsilon A \end{array} \right) \middle| \epsilon = \pm 1,\ A \in \mathrm{GL}(2, \mathbb{R}),\ \det(A) = \epsilon \right\} \subset G_1$$

is a (disconnected) 3-dimensional Lie subgroup. Let the restriction of $\pi : \mathcal{G}_1 \to M$ to $\mathcal{H}$ be denoted by the same symbol π.

One can verify that, restricted to $\mathcal{H}$, the 1-forms $\phi_7-\phi_3$, $\phi_6-\phi_2$, $\phi_5-\phi_1$ and ϕ_0 in the equation (5) become semi-basic relative to $\pi : \mathcal{H} \to U$. Hence, there exist functions Q_{ij} defined on $\mathcal{H}$ such that

$$\begin{aligned} &\phi_7 = \phi_3 + Q_{7i}\omega^i, \qquad \phi_6 = \phi_2 + Q_{6i}\omega^i, \\ &\phi_5 = \phi_1 + Q_{5i}\omega^i, \qquad \phi_0 = Q_{0i}\omega^i, \end{aligned} \tag{38}$$

where the summations are over $i = 0, 1, \ldots, 4$. There are ambiguities in these Q_{ij} as we can modify them without changing the form of the structure equation (5). Using such ambiguities, we can arrange that

$$Q_{71} = Q_{73} = Q_{62} = Q_{64} = Q_{51} = Q_{52} = Q_{53} = Q_{54} = 0; \tag{39}$$

the remaining Q_{ij} are then determined.

Applying $\mathrm{d}^2 = 0$ to (5) and reducing appropriately, we obtain

$$\begin{aligned} \mathrm{d}^2\omega^1 &\equiv \quad (Q_{63} - Q_{04})\omega^0 \wedge \omega^3 \wedge \omega^4 \quad \mathrm{mod}\ \omega^1, \omega^2, \\ \mathrm{d}^2\omega^2 &\equiv \quad (Q_{03} - Q_{74})\omega^0 \wedge \omega^3 \wedge \omega^4 \quad \mathrm{mod}\ \omega^1, \omega^2, \\ \mathrm{d}^2\omega^3 &\equiv \quad (Q_{02} + Q_{61})\omega^0 \wedge \omega^1 \wedge \omega^2 \quad \mathrm{mod}\ \omega^3, \omega^4, \\ \mathrm{d}^2\omega^4 &\equiv (-Q_{01} - Q_{72})\omega^0 \wedge \omega^1 \wedge \omega^2 \quad \mathrm{mod}\ \omega^3, \omega^4. \end{aligned}$$

This implies that

$$Q_{61} = -Q_{02}, \quad Q_{63} = Q_{04}, \quad Q_{72} = -Q_{01}, \quad Q_{74} = Q_{03}.$$

Now all coefficients in the structure equation (5) are expressed in terms of Q_{0i} ($i = 0, 1, \ldots, 4$) and Q_{j0} ($j = 5, 6, 7$). By applying $\mathrm{d}^2 = 0$ to (5), it is not difficult to verify that, reduced modulo $\omega^0, \omega^1, \ldots, \omega^4$, the following congruences hold:

$$\begin{aligned} &\mathrm{d}\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix} \equiv \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & -\phi_1 \end{pmatrix}\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}, \quad \mathrm{d}Q_{00} \equiv 0, \\ &\mathrm{d}\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix} \equiv \begin{pmatrix} 0 & \phi_3 & -\phi_2 \\ 2\phi_2 & -2\phi_1 & 0 \\ -2\phi_3 & 0 & 2\phi_1 \end{pmatrix}\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}. \end{aligned} \tag{40}$$

The congruences (40) tell us how the remaining Q_{ij} transform under the action by the identity component of H. Moreover, it is easy to compute directly from (5) to verify that

$$\begin{aligned} &Q_{00}(u \cdot h_0) = -Q_{00}(u), \\ &\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}(u \cdot h_0) = \begin{pmatrix} -Q_{01} & Q_{03} \\ Q_{02} & -Q_{04} \end{pmatrix}(u), \\ &\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}(u \cdot h_0) = \begin{pmatrix} -Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}(u), \\ &h_0 = \mathrm{diag}(-1, -1, 1, 1, -1) \in H \end{aligned} \tag{41}$$

hold for any $u \in \mathcal{H}$.

Note that H is generated by its identity component and h_0. Combining (40) and (41), it is easy to see that $Q_{01}Q_{04} - Q_{02}Q_{03}$ and $|Q_{00}|$ are local invariants of the underlying Euler-Lagrange system.

Moreover, using (40) and (41), it is easy to see that the H-orbit of

$$q(u) := \begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}(u), \quad u \in \mathcal{H}$$

consists of all 2-by-2 matrices with the same determinant as $q(u)$. Now we are ready to prove the following lemma.

Lemma B.1. *If* $\det(q) \neq 0$ *on* $\mathcal{H}$*, then there is a canonical way to define a coframing on* U*.*

Proof. If the function $L := \det(q)$ is nonvanishing on U, one can reduce to the subbundle $\mathcal{H}_1$ of $\mathcal{H}$ defined by $q = \mathrm{diag}(L, 1)$. It is easy to see that each fiber of $\mathcal{H}_1$ over U contains a single element. □

Remark 6. As a result of Lemma B.1, if $\det(q) \neq 0$ on U, then the corresponding hyperbolic Euler-Lagrange system has a symmetry of dimension at most 5. This is a consequence of applying the *Frobenius Theorem*.

Now we proceed to complete the proof of Proposition 6.2. Recall that the coframing $(\eta^0, \eta^1, \dots, \eta^4)$ and the ϕ_α in (28) verify the equation (5), $S_1 = \mathrm{diag}(1, 1)$, and $S_2 = \mathbf{0}$. Moreover, we have chosen the ϕ_α to satisfy (39), where Q_{ij} are computed using (38). By (28), it is immediate that

$$Q_{00} = Q_{02} = 0, \quad Q_{01} = -Q_{04} = \frac{1}{\sqrt{2}}, \quad Q_{03} = 1,$$
$$Q_{70} = 1, \quad Q_{60} = -1, \quad Q_{50} = \sqrt{2}.$$

Clearly, $\det(q) = Q_{01}Q_{04} - Q_{02}Q_{03} = -1/2 \neq 0$. By Lemma B.1 and Remark 6, the hyperbolic Euler-Lagrange system considered in Proposition 6.2 has a symmetry of dimension at most 5. Because that Euler-Lagrange system is homogeneous, it follows that its symmetry has dimension 5.

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