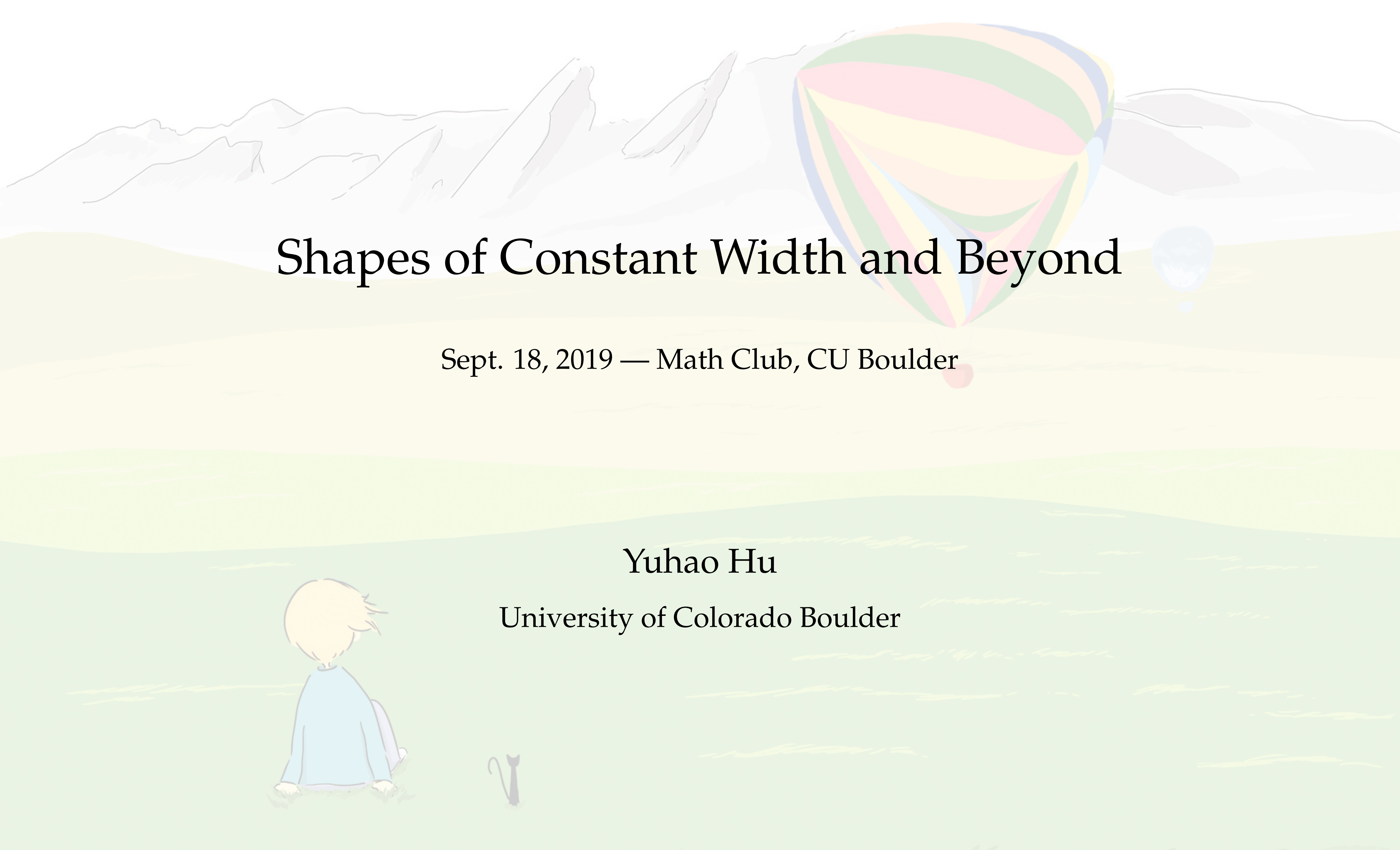


Shapes of Constant Width and Beyond

Sept. 18, 2019 — Math Club, CU Boulder

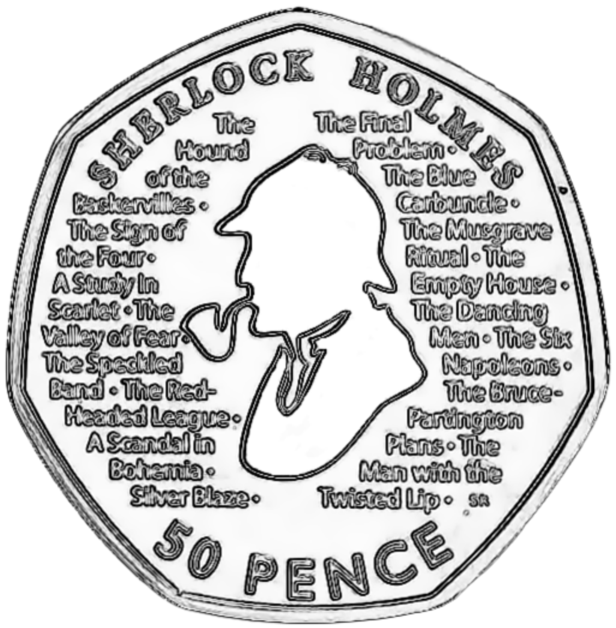
Yuhao Hu

University of Colorado Boulder

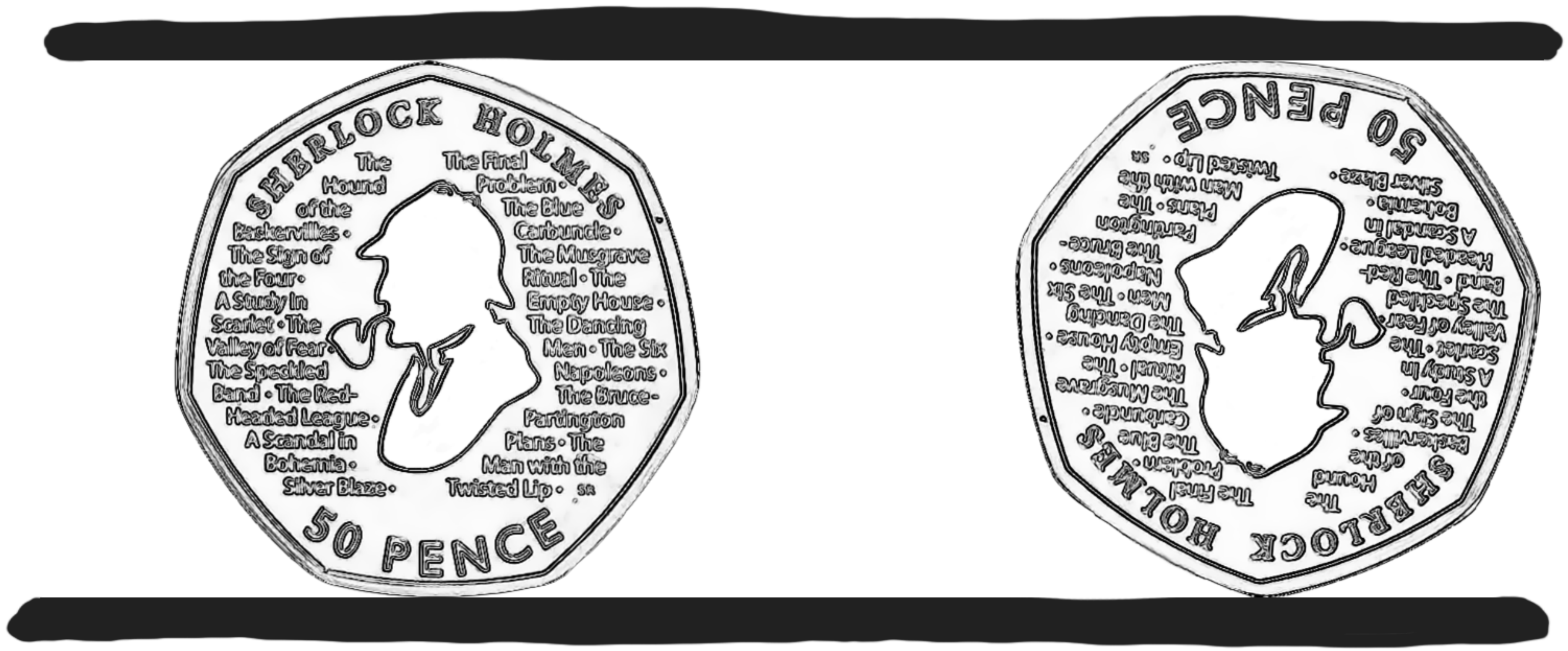
A. 50 Pence


SHERLOCK HOLMES
The Hound of the Baskervilles • The Sign of the Four • A Study in Scarlet • The Valley of Fear • The Speckled Band • The Red-Headed League • A Scandal in Bohemia • Silver Blaze •
The Final Problem • The Blue Carbuncle • The Musgrave Ritual • The Empty House • The Dancing Men • The Six Napoleons • The Bruce-Partington Plans • The Man with the Twisted Lip •
50 PENCE

A. 50 Pence


SHERLOCK HOLMES
The Hound of the Baskervilles • The Sign of the Four • A Study in Scarlet • The Valley of Fear • The Speckled Band • The Red-Headed League • A Scandal in Bohemia • Silver Blaze •
The Final Problem • The Blue Carbuncle • The Musgrave Ritual • The Empty House • The Dancing Men • The Six Napoleons • The Bruce-Partington Plans • The Man with the Twisted Lip •
SR
50 PENCE

A. 50 Pence


SHERLOCK HOLMES
The Hound of the Baskervilles • The Sign of the Four • A Study in Scarlet • The Valley of Fear • The Speckled Band • The Red-Headed League • A Scandal in Bohemia • Silver Blaze •
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SR
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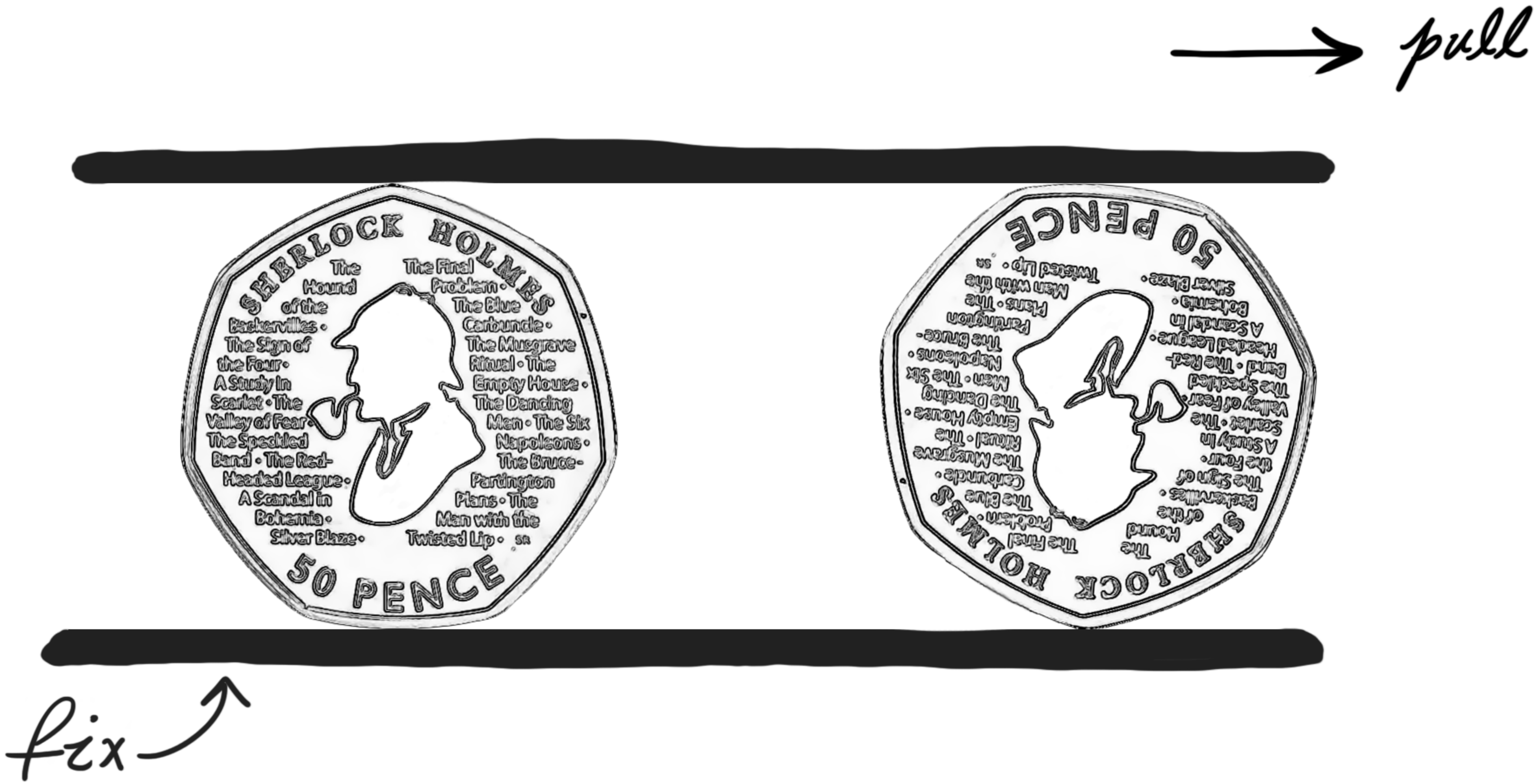
A. 50 Pence


pull
SHERLOCK HOLMES
The Hound of the Baskervilles • The Sign of the Four • A Study in Scarlet • The Valley of Fear • The Speckled Band • The Red-Headed League • A Scandal in Bohemia • Silver Blaze •
The Final Problem • The Blue Carbuncle • The Musgrave Ritual • The Empty House • The Dancing Men • The Six Napoleons • The Bruce-Partington Plans • The Man with the Twisted Lip •
50 PENCE
fix

B. Manhole Cover

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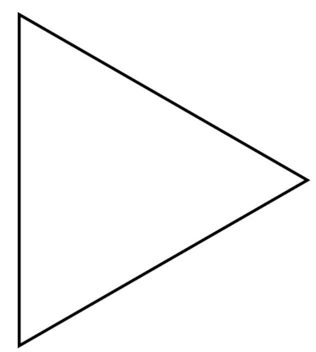
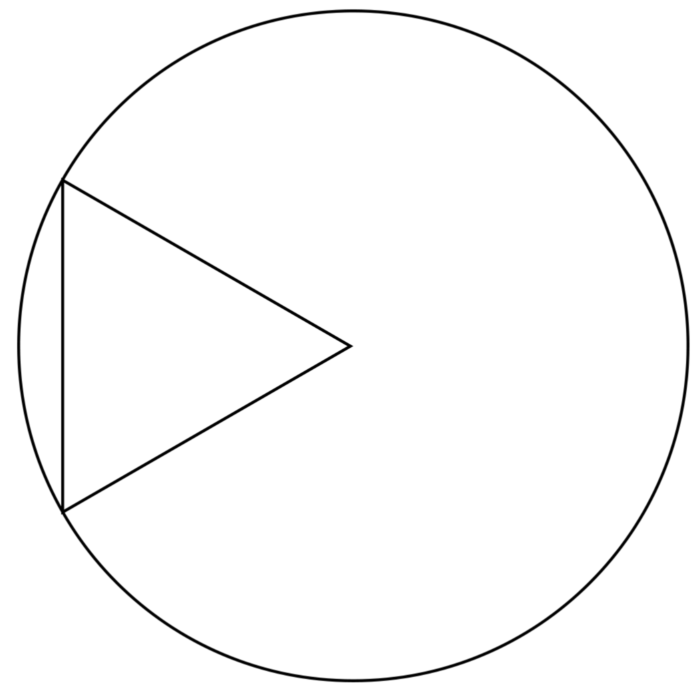
C. Ruler-Compass Construction (Starting with a regular $(2n+1)$-gon . . .)



C. Ruler-Compass Construction (Starting with a regular $(2n+1)$-gon . . .)



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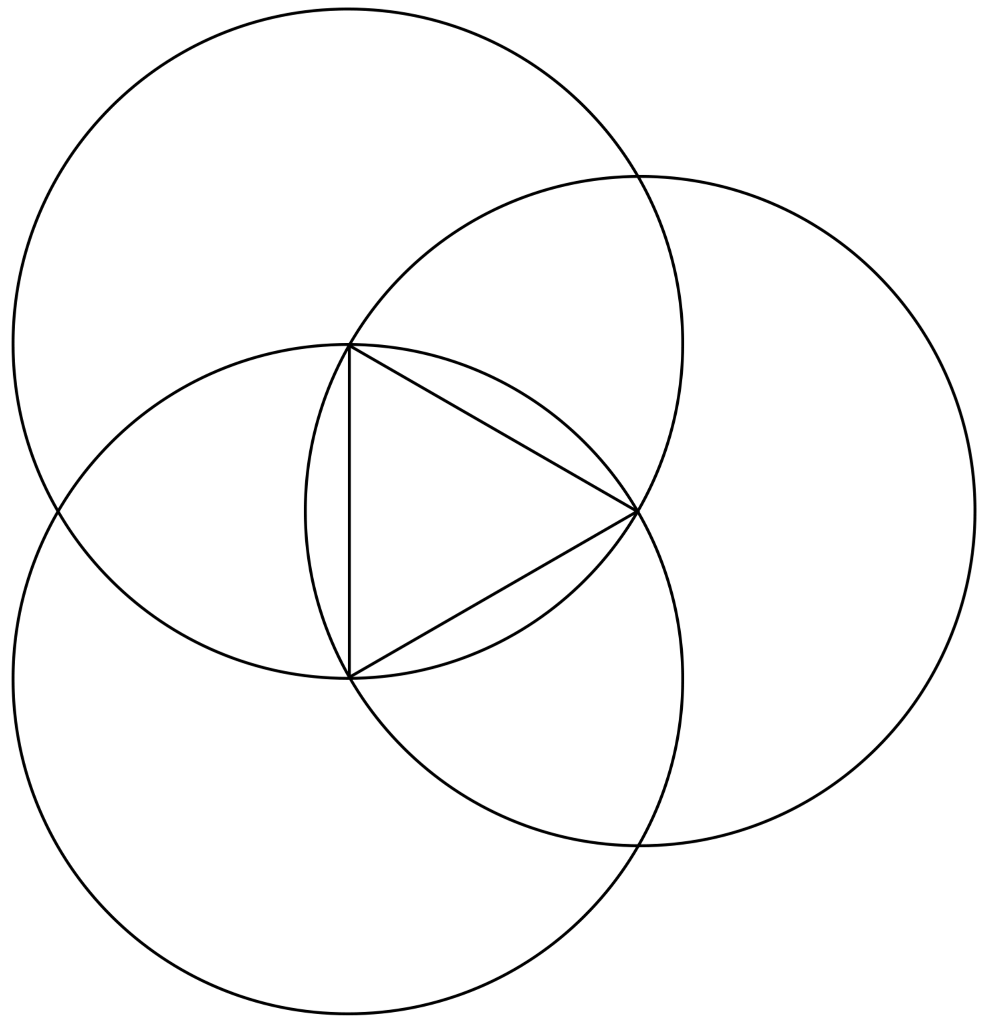

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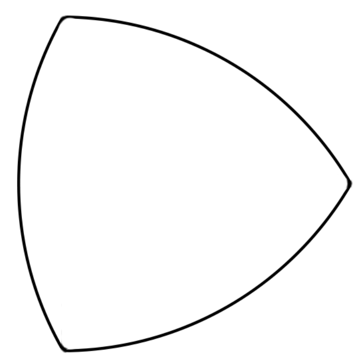
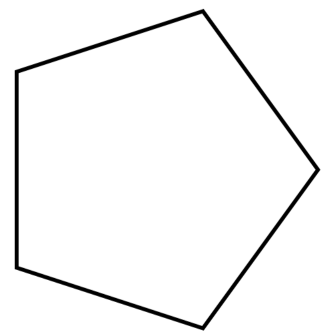
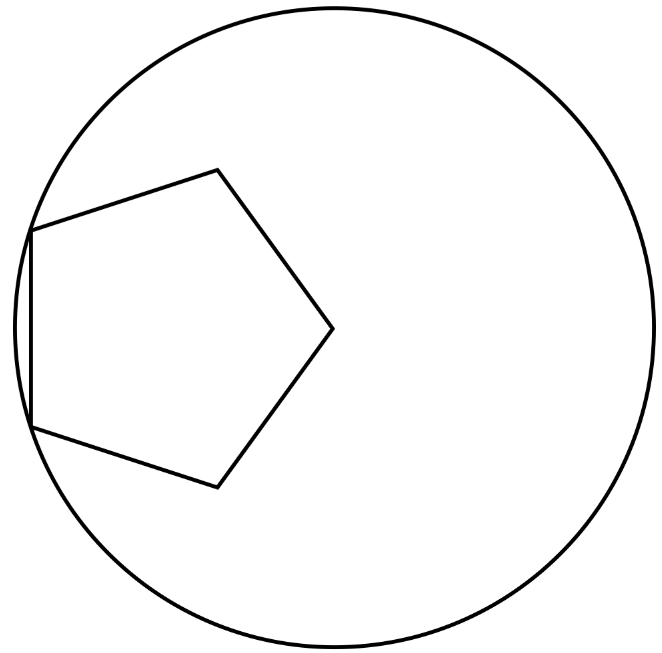


Reuleaux Triangle

C. Ruler-Compass Construction (Starting with a regular $(2n+1)$-gon . . .)



C. Ruler-Compass Construction (Starting with a regular $(2n+1)$-gon ...)



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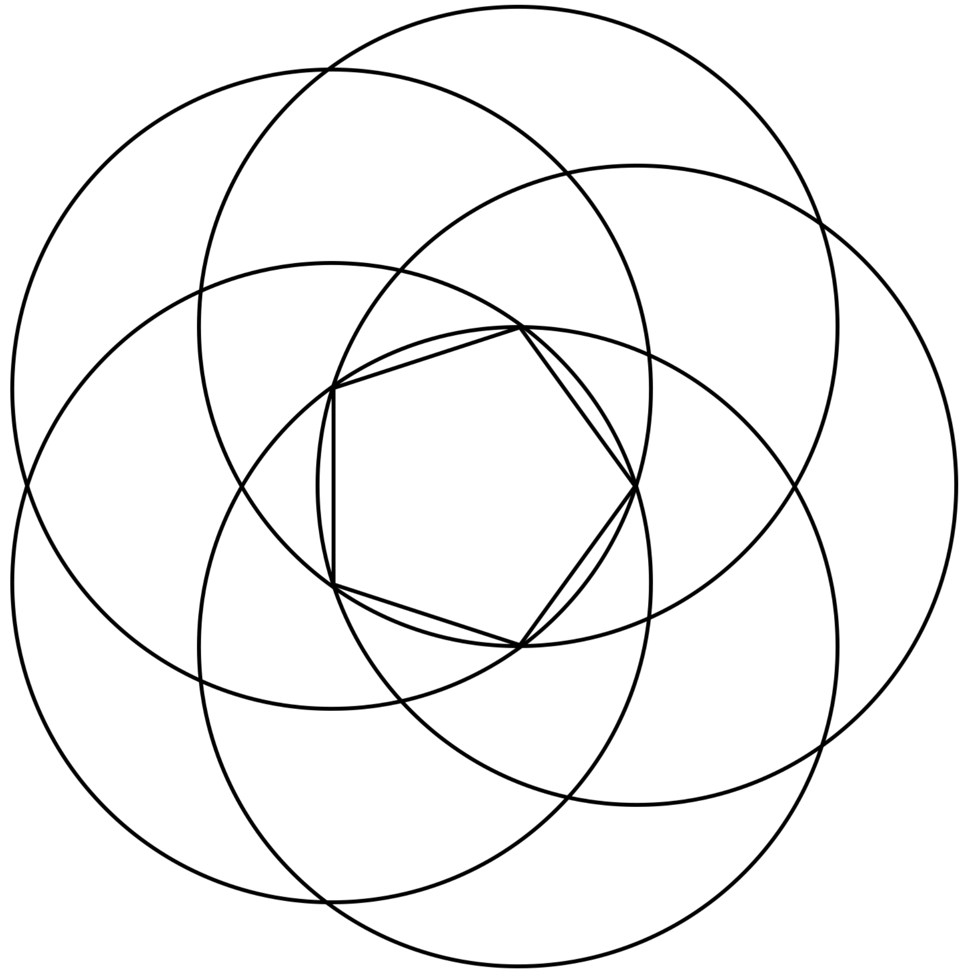

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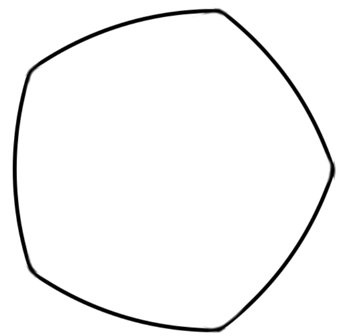

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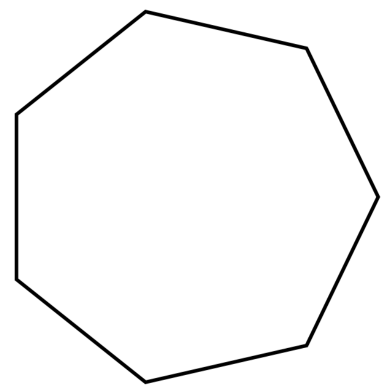

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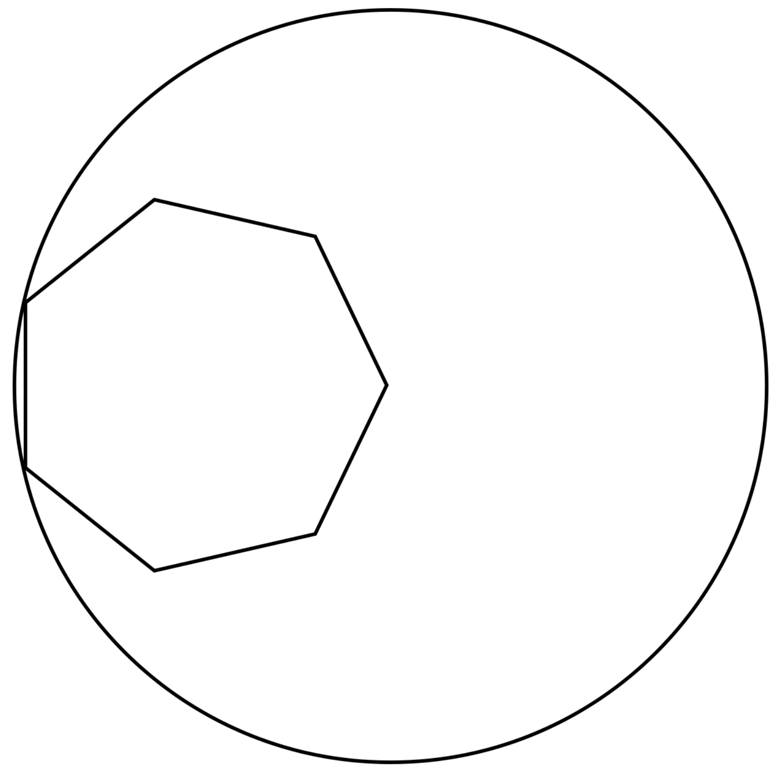

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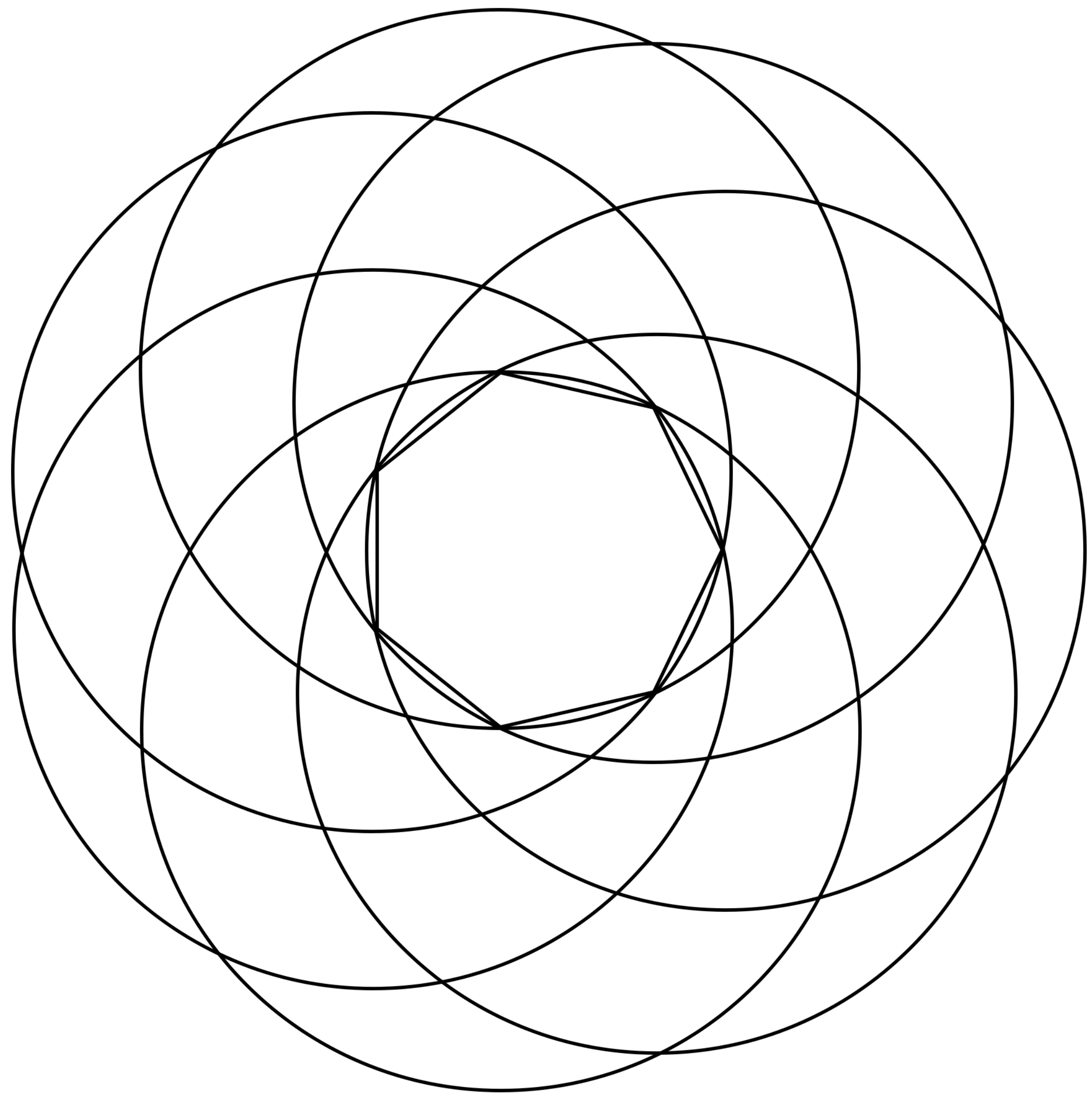

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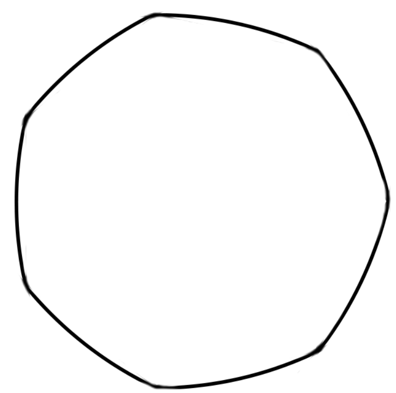
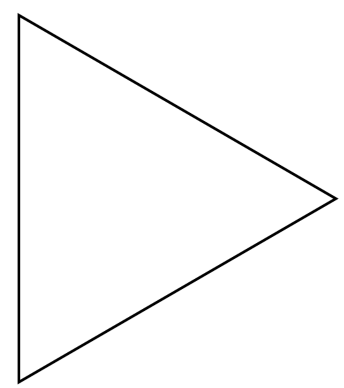


D. Ruler-Compass Construction (Removing corners …)



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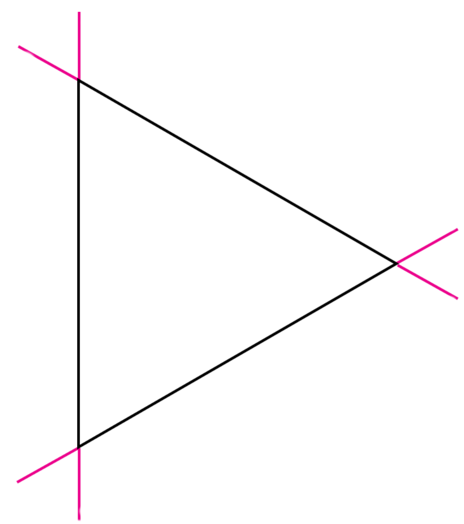

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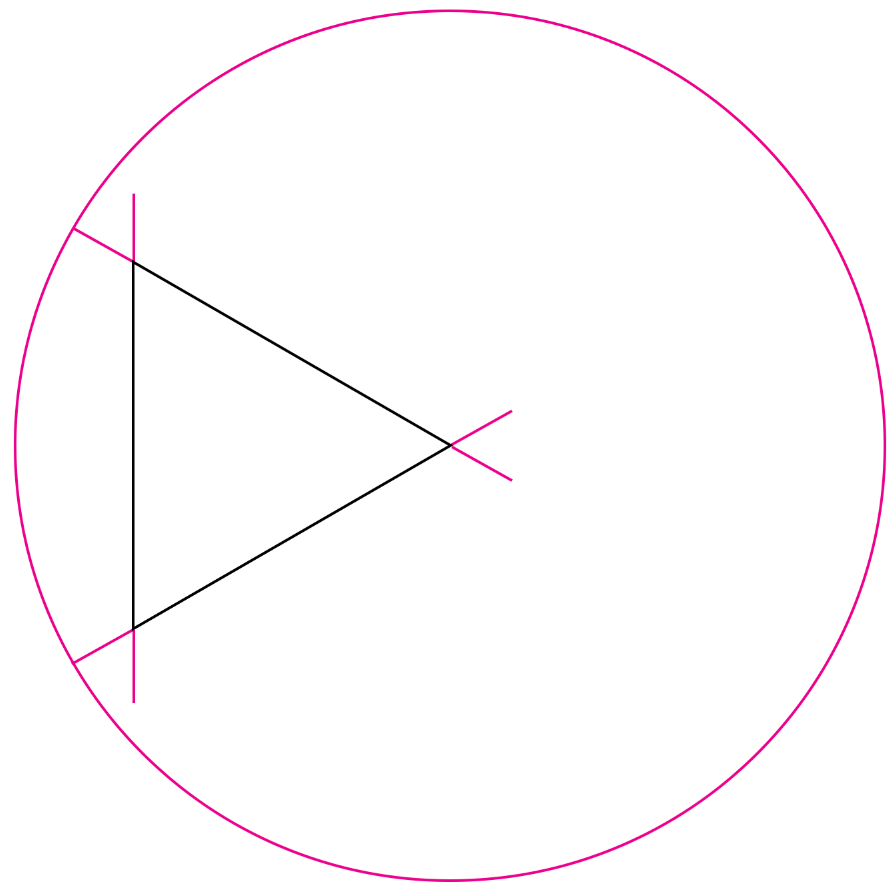

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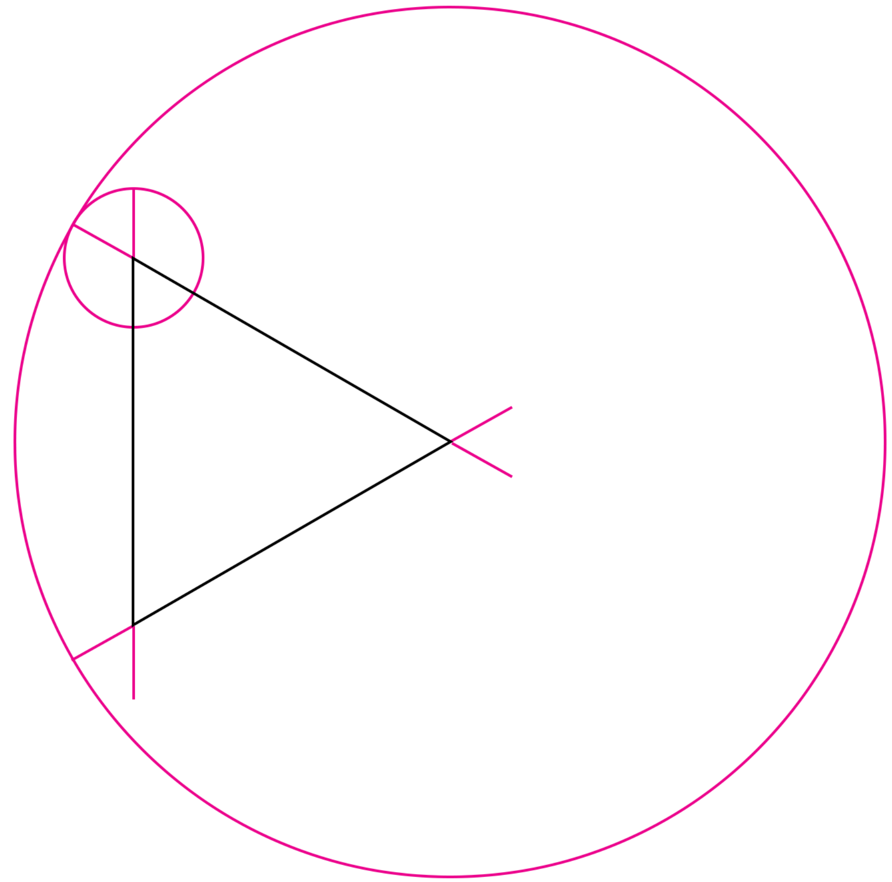

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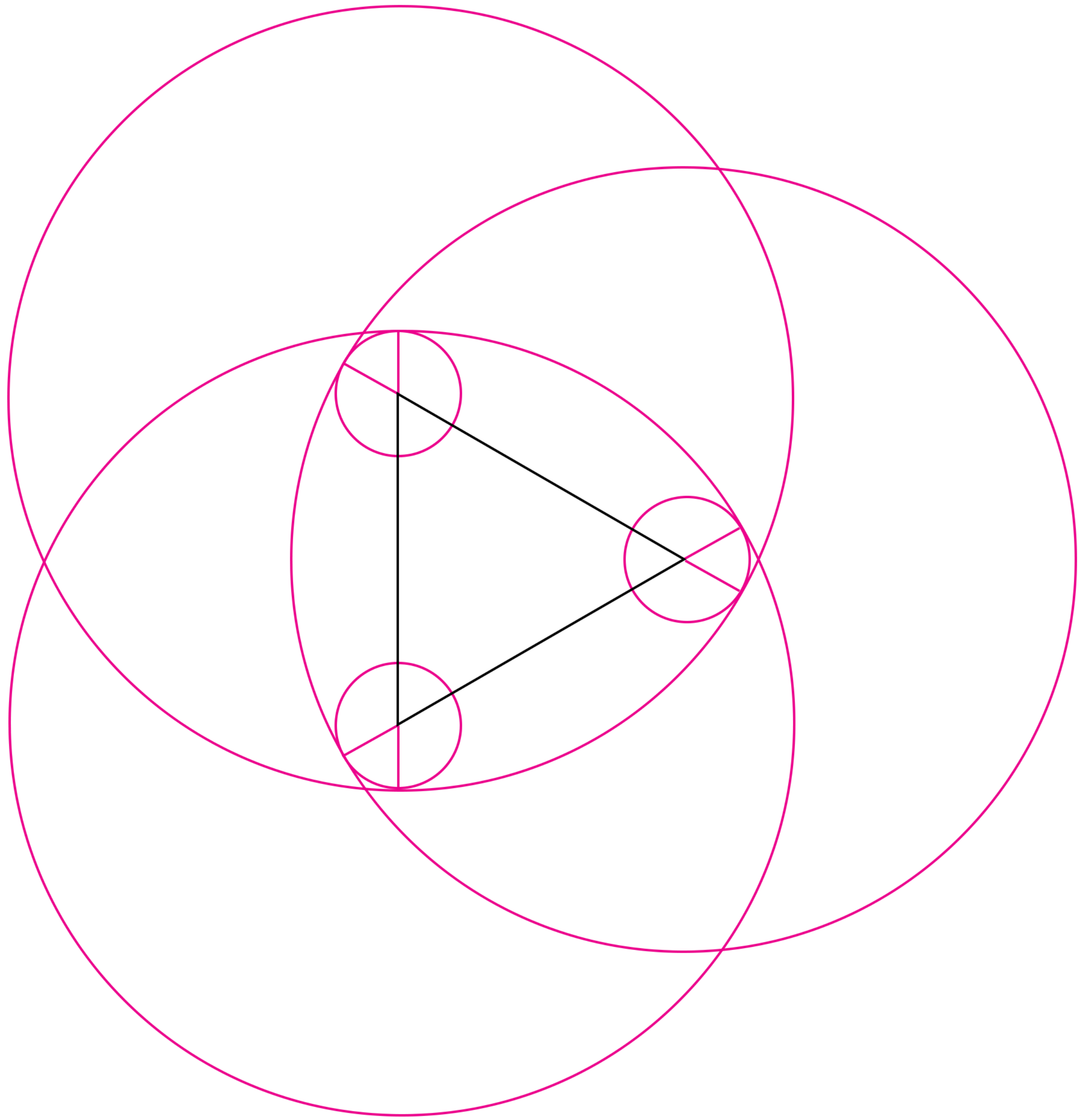

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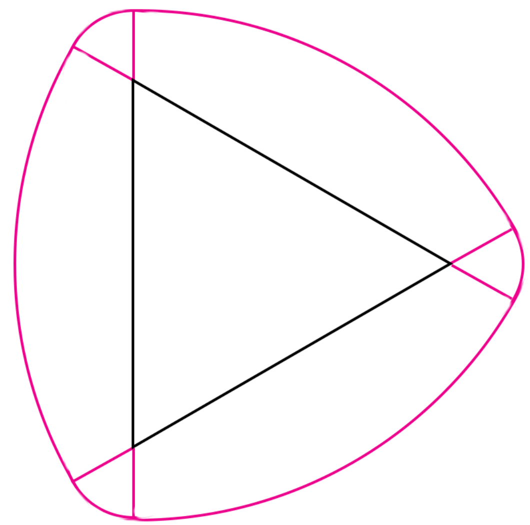
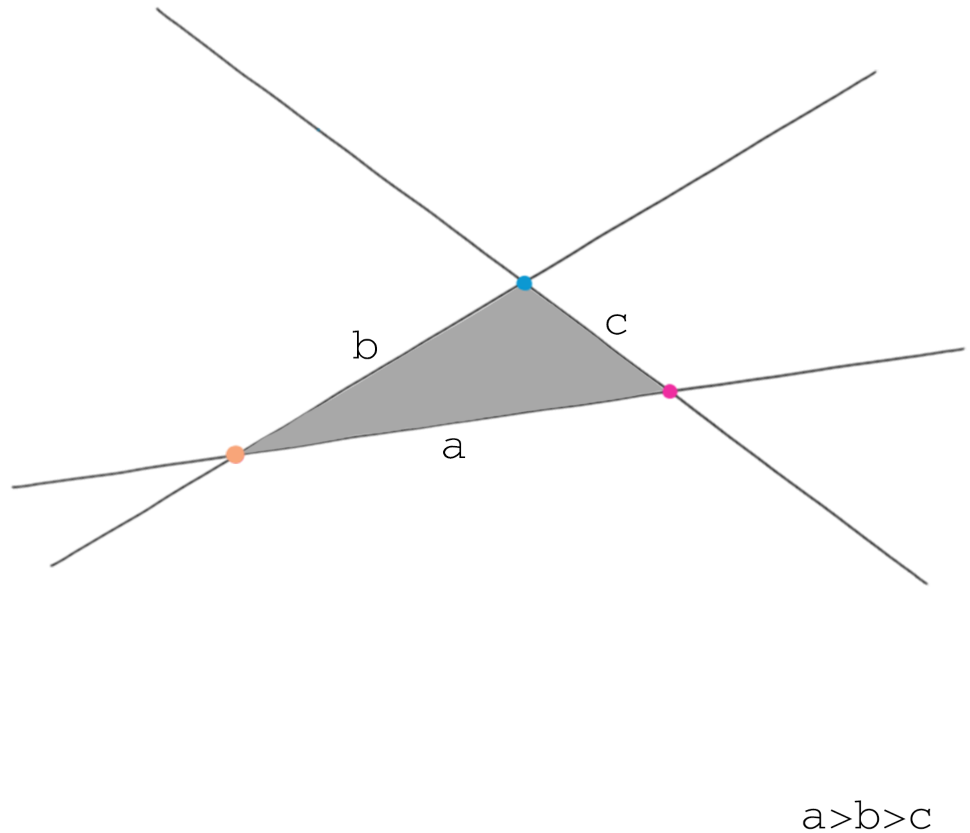


E. Ruler-Compass Construction (From any triangle with sides $a > b > c \ldots$)



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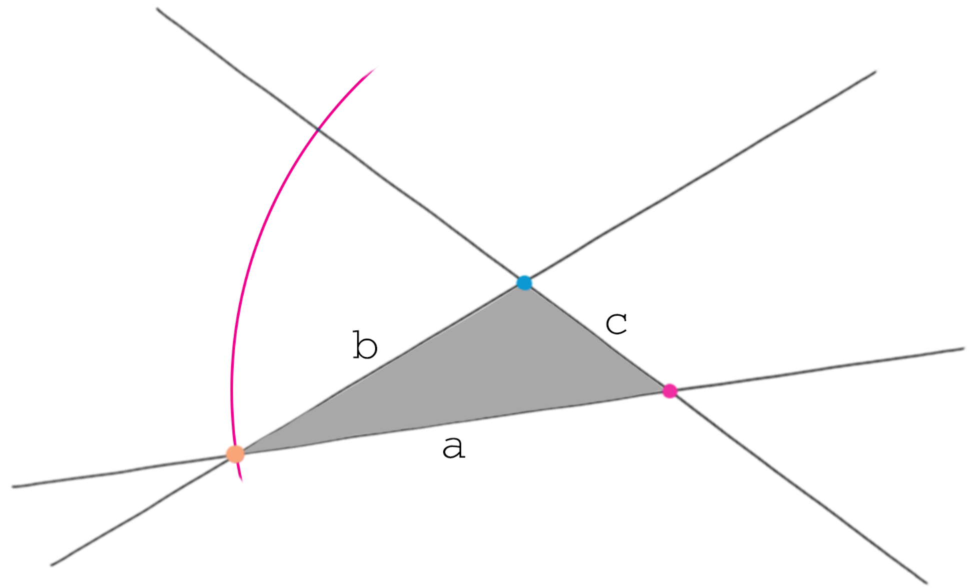

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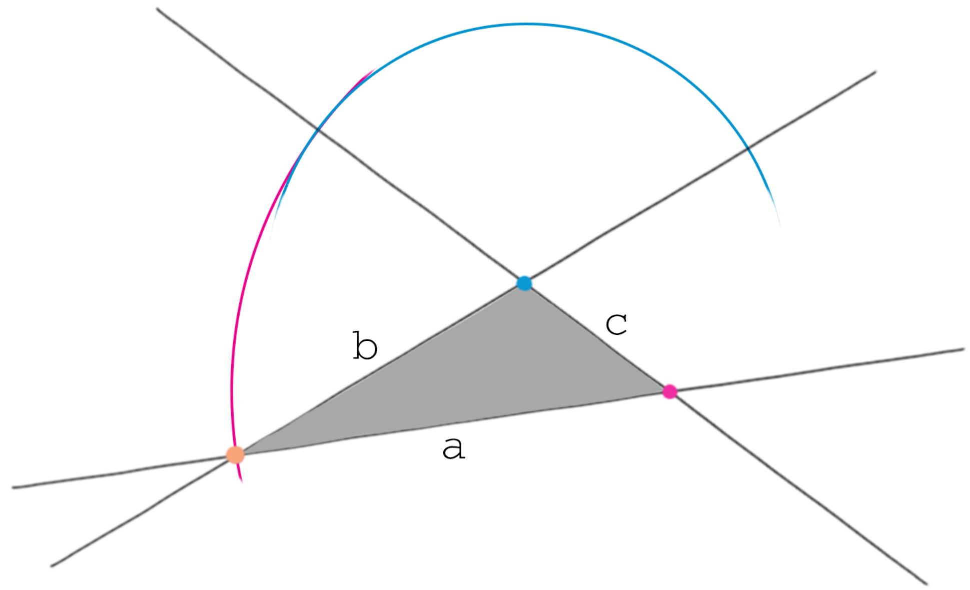

a>b>c

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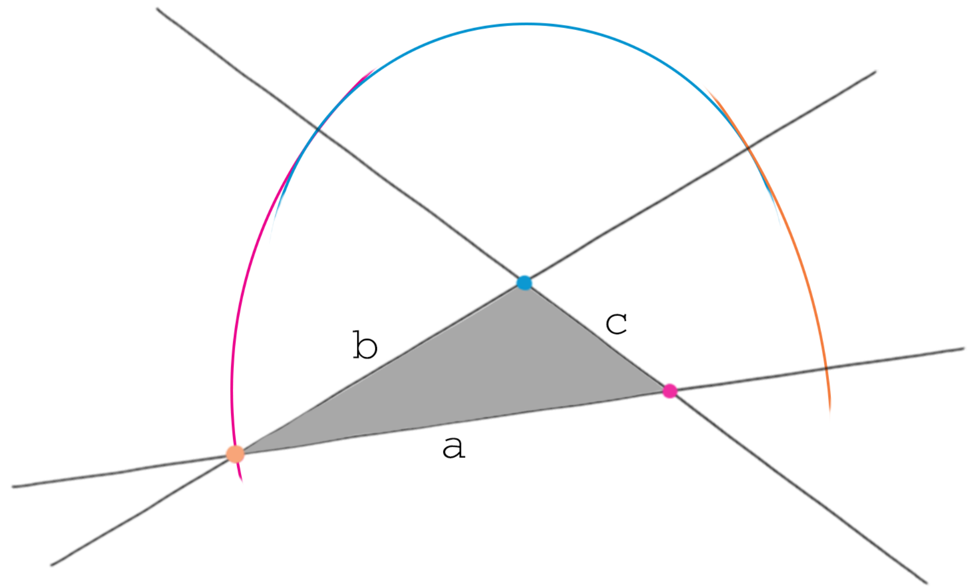

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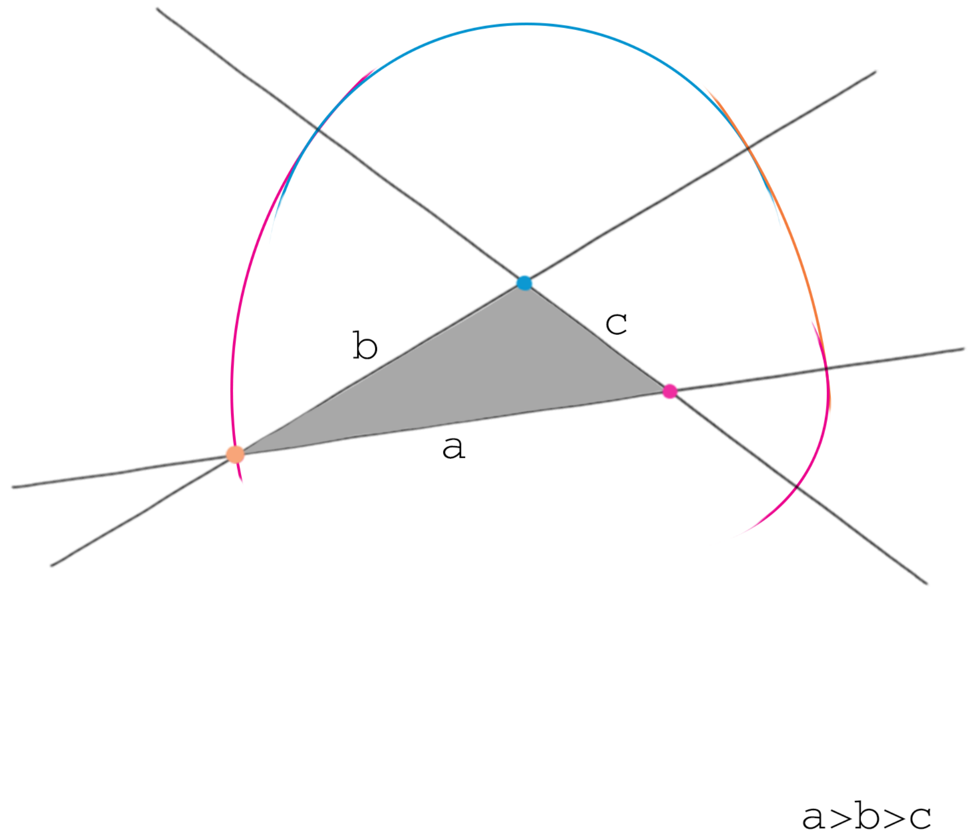

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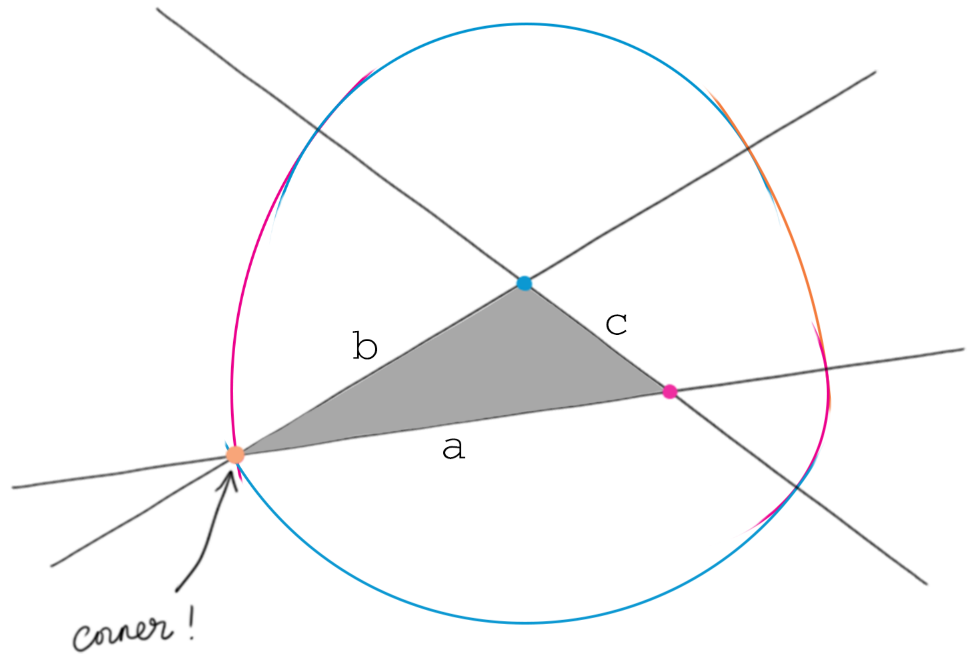

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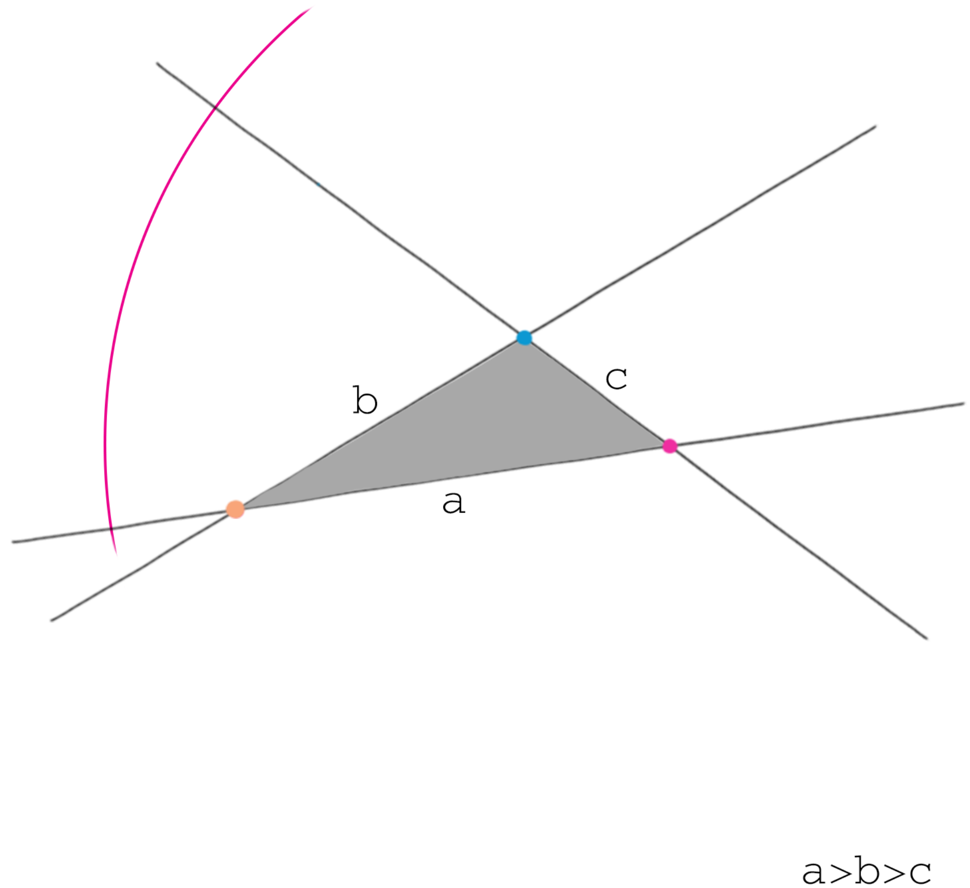

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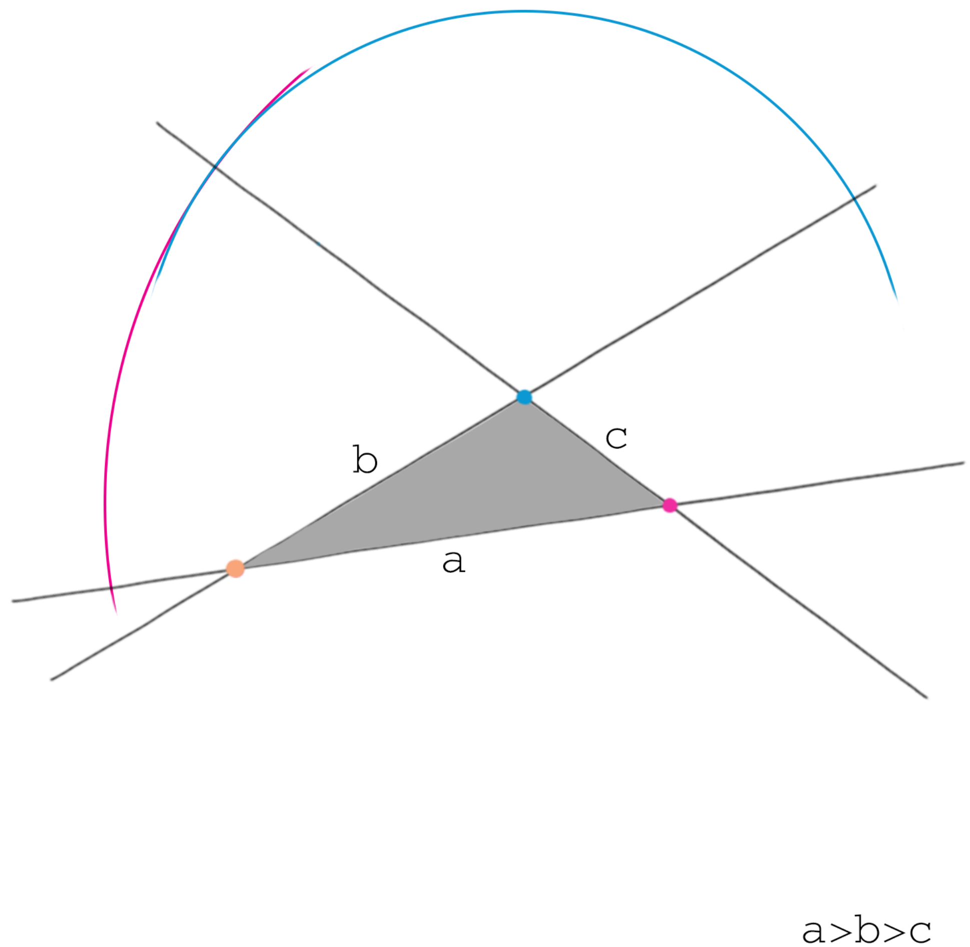

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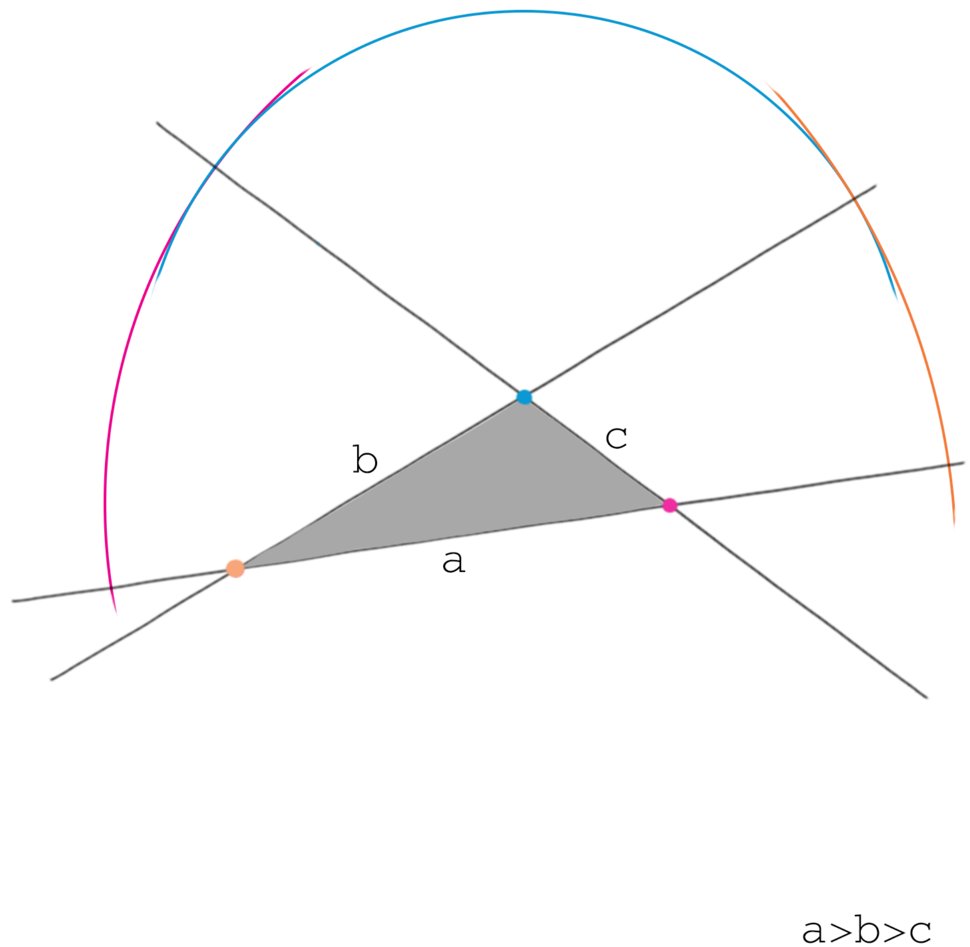

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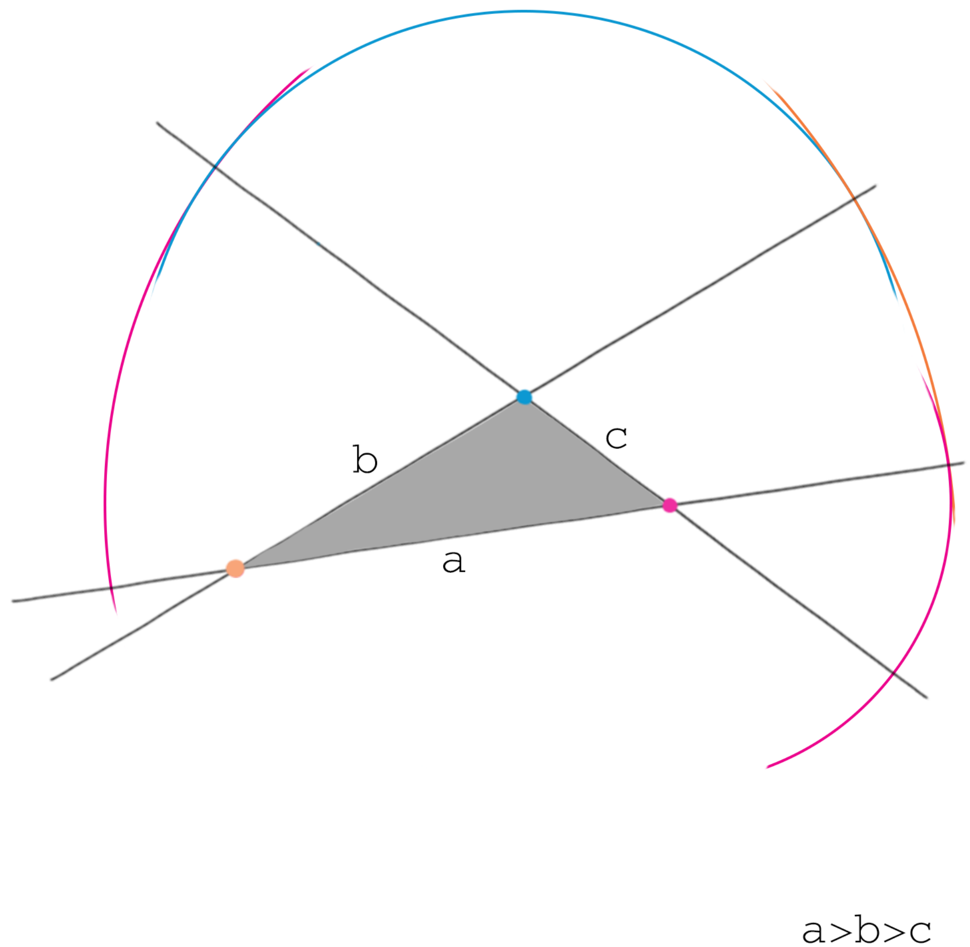

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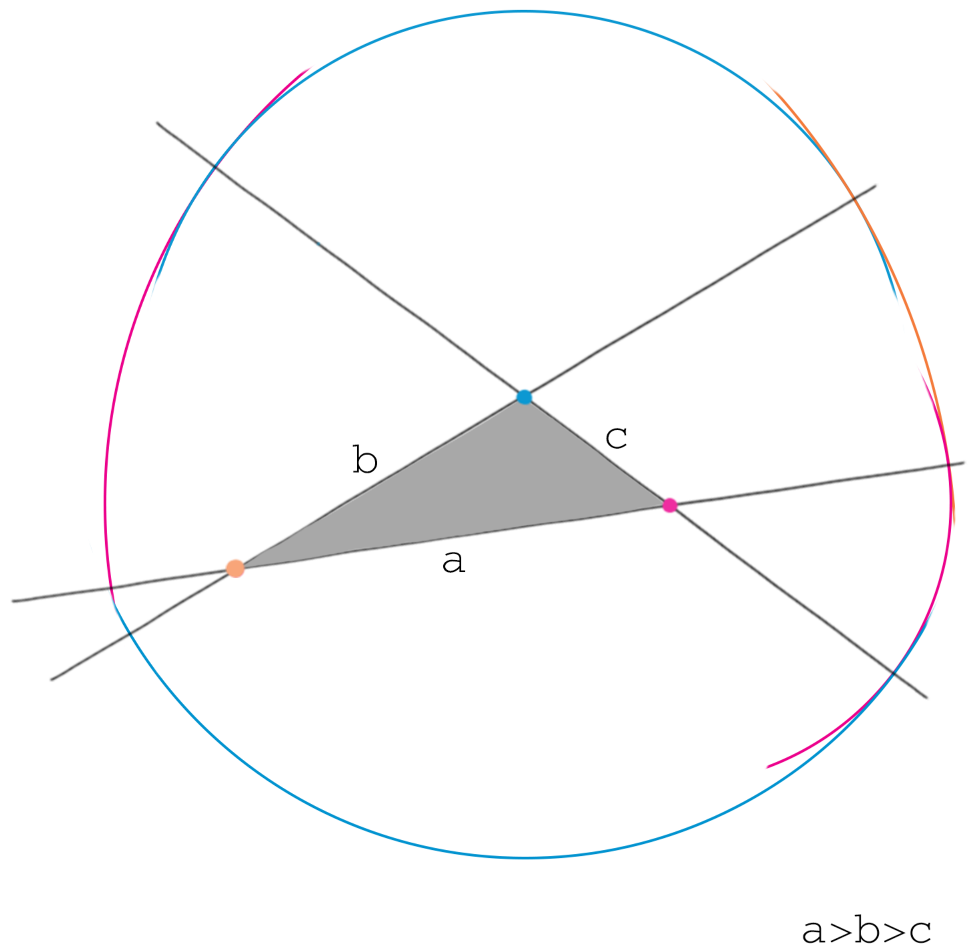

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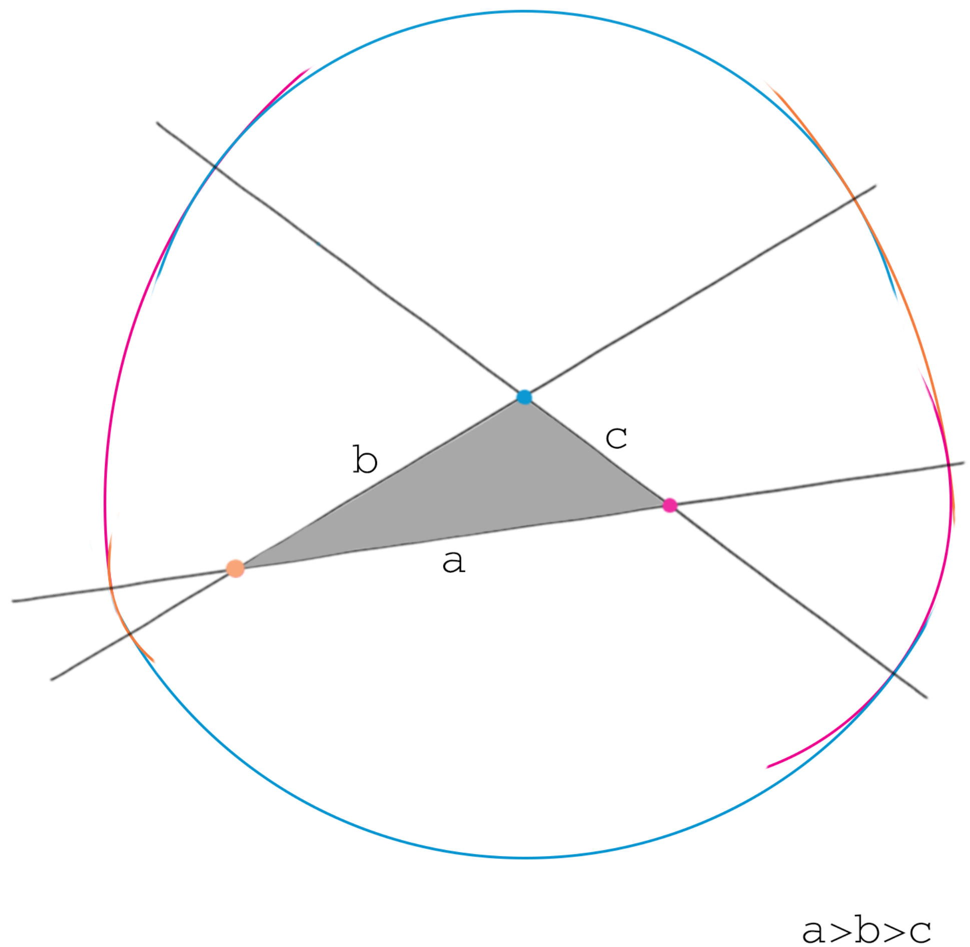

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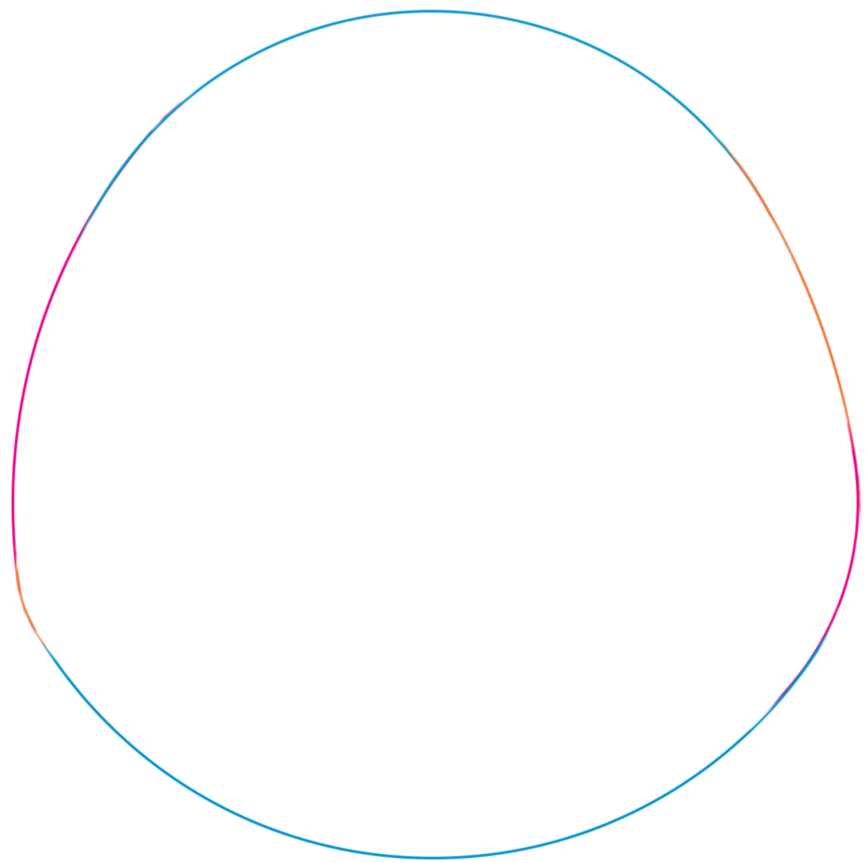

F. Strictly Convex Regions in $\mathbb{R}^2$

Definition. Given a region $\mathcal{K} \subset \mathbb{R}^2$, it is said to be **convex** if the line segment connecting any two points $p, q \in \mathcal{K}$ remains entirely in $\mathcal{K}$, and it is said to be **strictly convex** if, for any two points $p, q \in \mathcal{K}$ the interior of the line segment connecting p, q lives entirely in the interior of $\mathcal{K}$.

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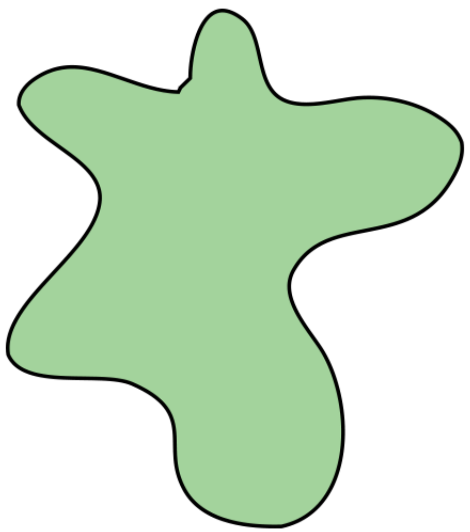

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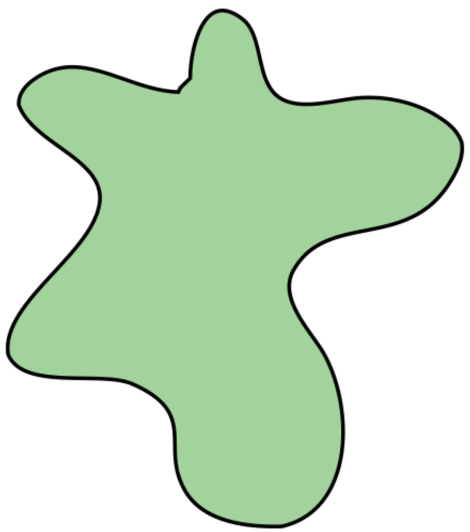

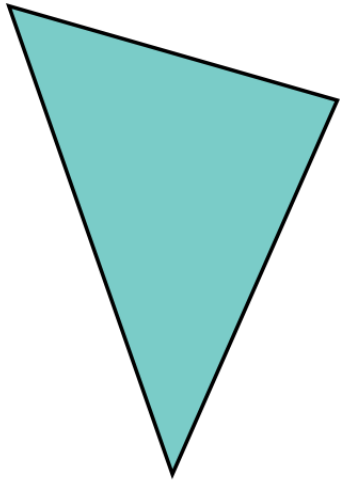

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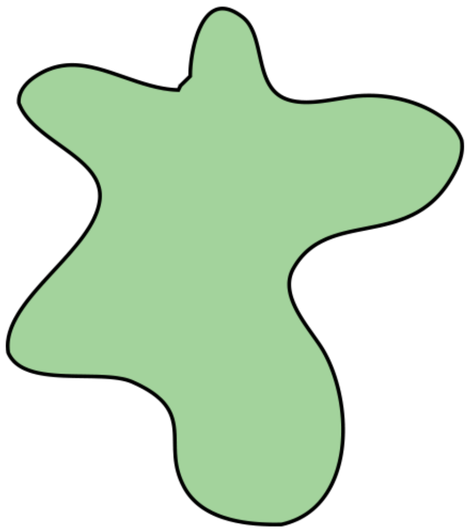
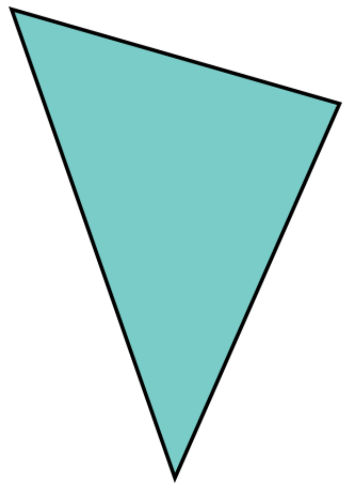
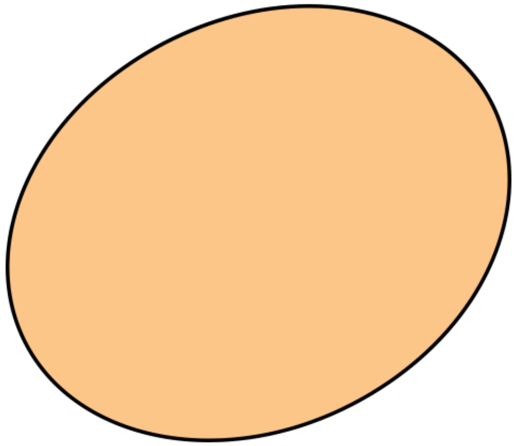
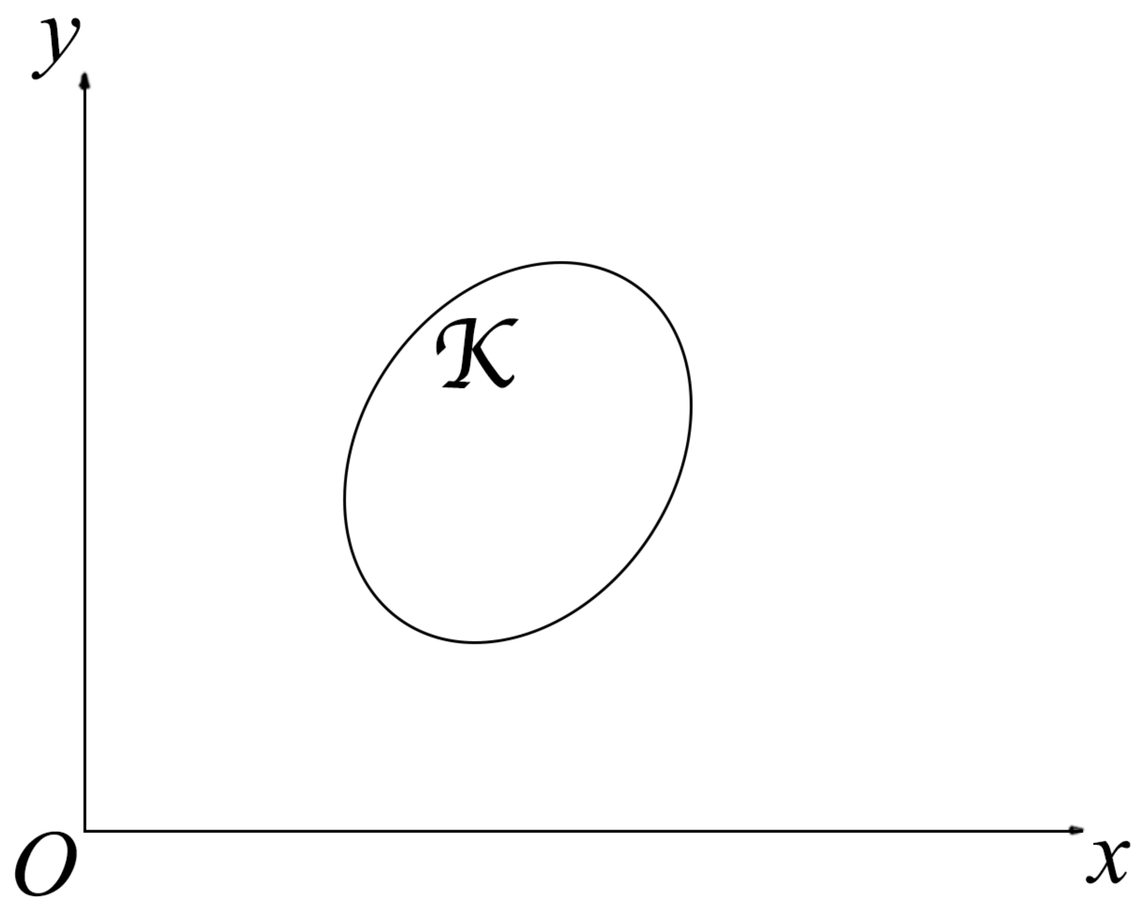






G. The Support Function $p(\theta)$



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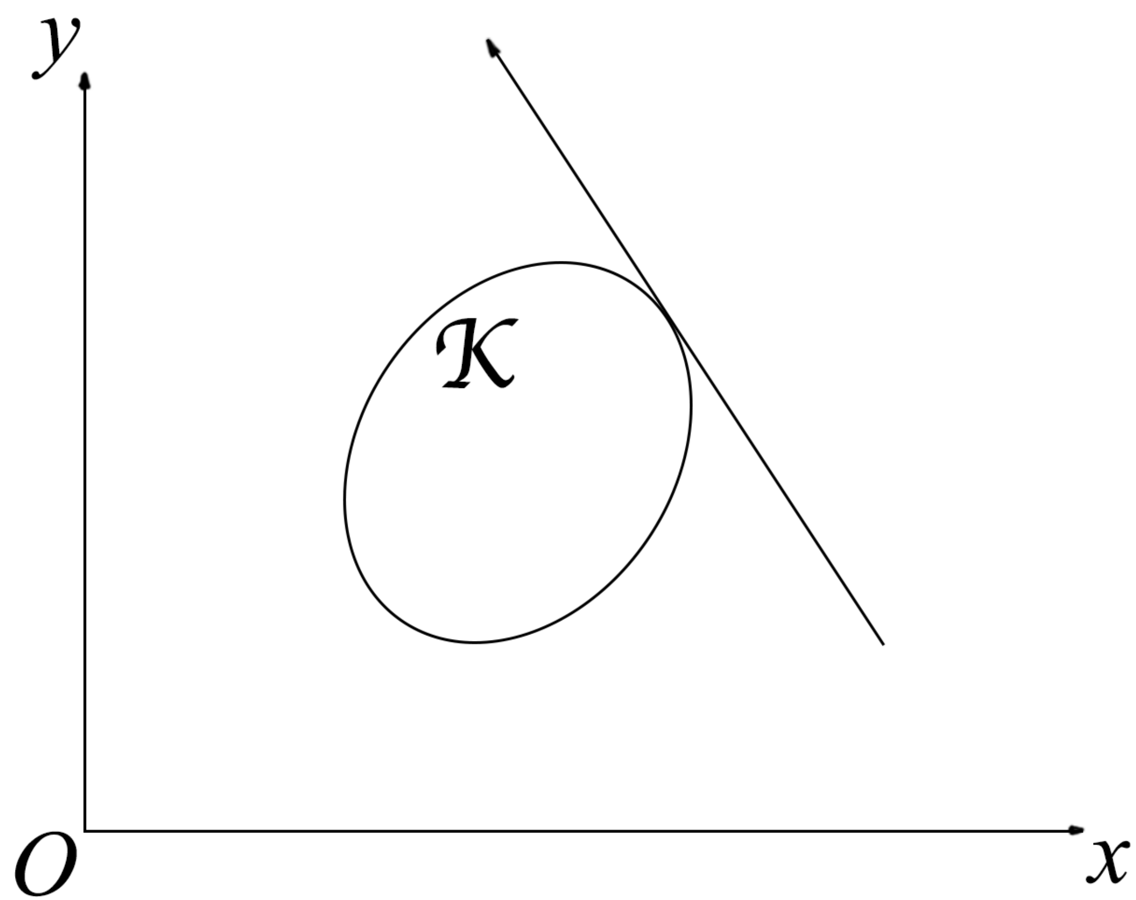

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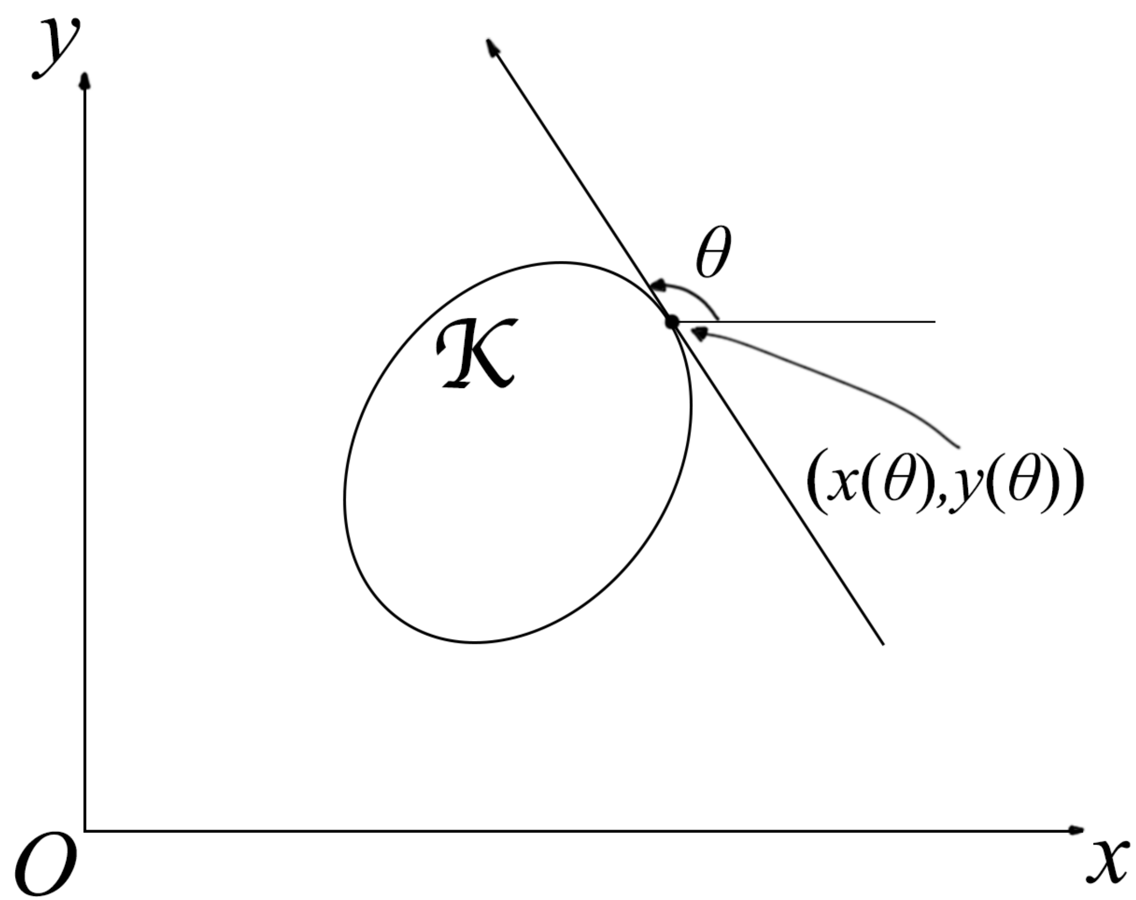

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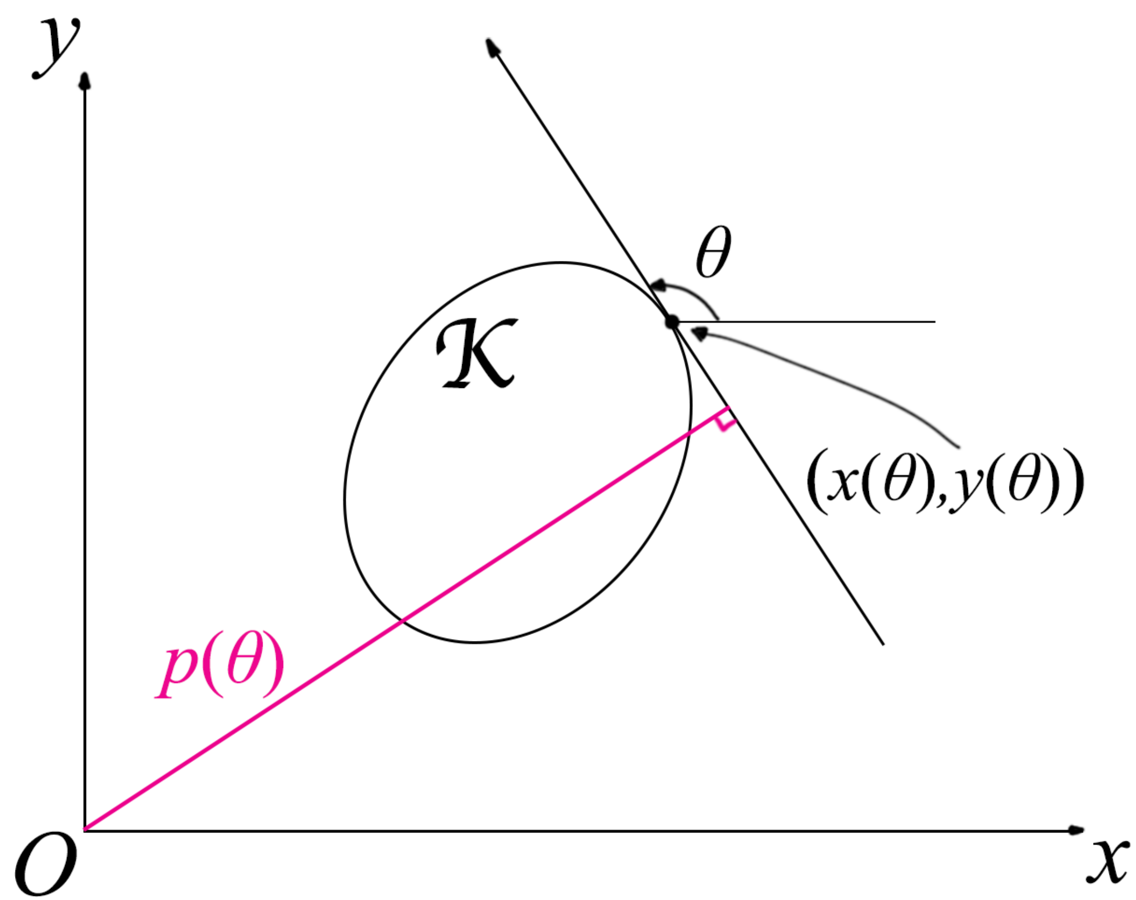

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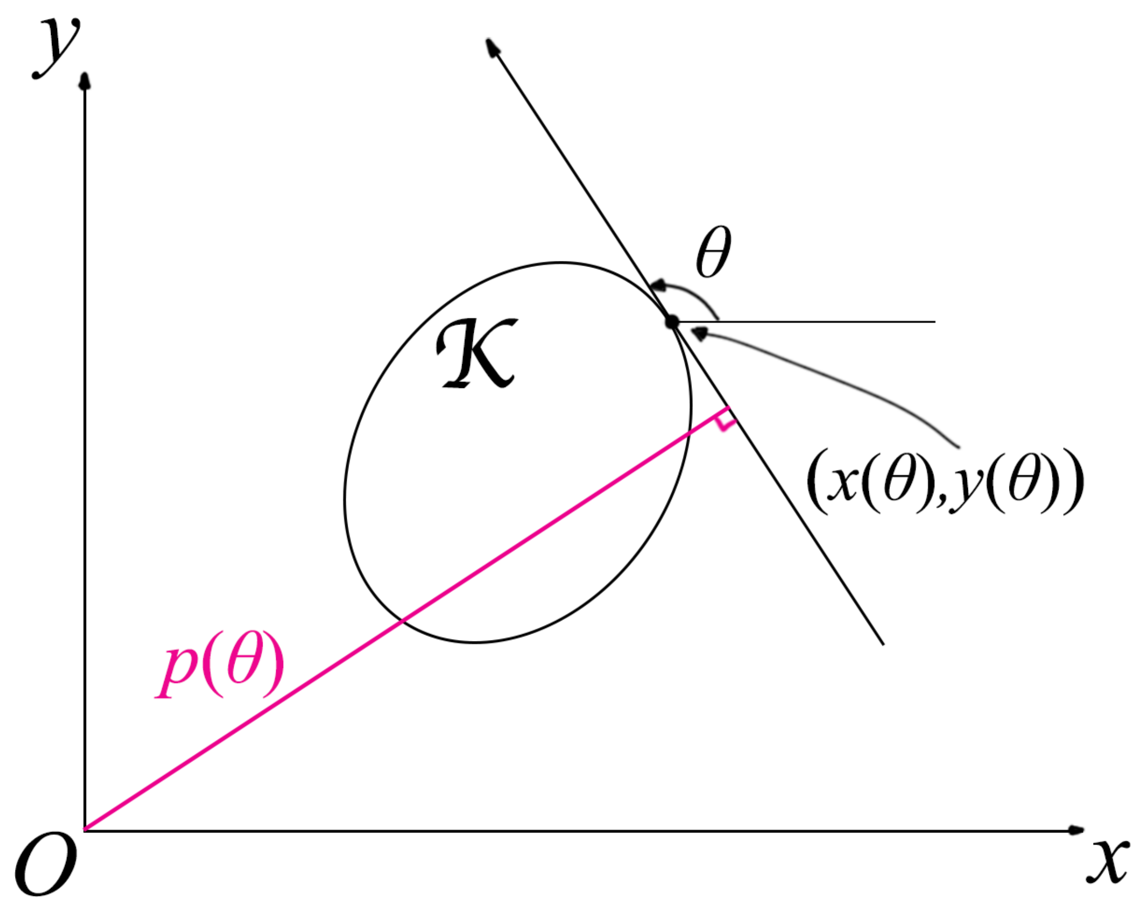

$$p(\theta) = \big(x(\theta), y(\theta)\big) \cdot (\sin\theta, -\cos\theta)$$
$$= x(\theta)\sin\theta - y(\theta)\cos\theta.$$

H. From $p(\theta)$ to $(x(\theta), y(\theta))$

Theorem. Given a (closed, bounded) strictly convex region $\mathcal{K}$, the support function $p(\theta)$ is C^1 (i.e. continuously differentiable). Moreover, we have

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} p(\theta) \\ p'(\theta) \end{pmatrix}.$$

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Idea: Let

$$\mathbf{x}(\theta) := \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}, \qquad \mathbf{u} := \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

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Since $\mathbf{x}'$ is perpendicular to $\mathbf{u}$, $\mathbf{u}^T\mathbf{x}' = 0$, and we obtain

$$\begin{pmatrix} p \\ p' \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T \\ \mathbf{u}'^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \mathbf{x}.$$

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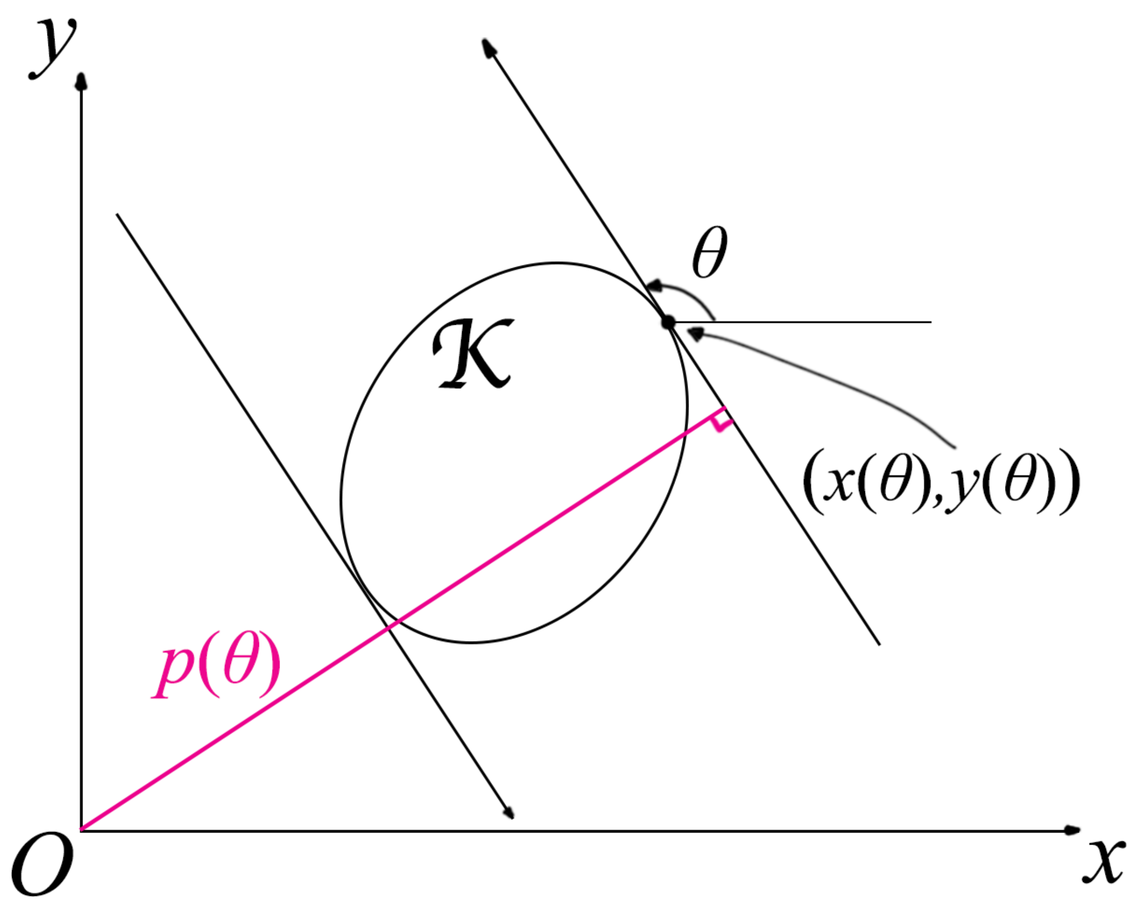
$$\begin{pmatrix} p \\ p' \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T \\ \mathbf{u}'^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \mathbf{x}.$$

Now solve this linear system for $\mathbf{x}$.

I. The Width Function $W(\theta)$



$$p(\theta) = x(\theta)\sin\theta - y(\theta)\cos\theta, \qquad W(\theta) = p(\theta) + p(\theta + \pi).$$

J. Using Fourier Series

Idea: To obtain curves of constant width, first find a C^1, 2π-periodic function $p(\theta)$ that satisfies

$$p(\theta) + p(\theta + \pi) = D$$

for some constant (diameter) $D > 0$, then use the formula

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Fourier expansion of a C^1, 2π-periodic function $p(\theta)$:

$$p(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

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It follows that $a_{2k}, b_{2k} = 0, \quad k = 1, 2, \ldots$

In other words, *for $p(\theta)$ to be the support function of a curve of constant width, its Fourier series can only contain the odd terms and the constant. Moreover, $a_0 = D$*.

K. Plotting CCW

Note. Adding a linear combination of $\sin\theta$ and $\cos\theta$ to $p(\theta)$ would result in a shifting of the shape. Shifting the variable θ by a constant would result in a rotation of the shape. Hence, the simplest, non-circular case would be when

$$p(\theta) = \frac{D}{2} + \cos 3\theta.$$

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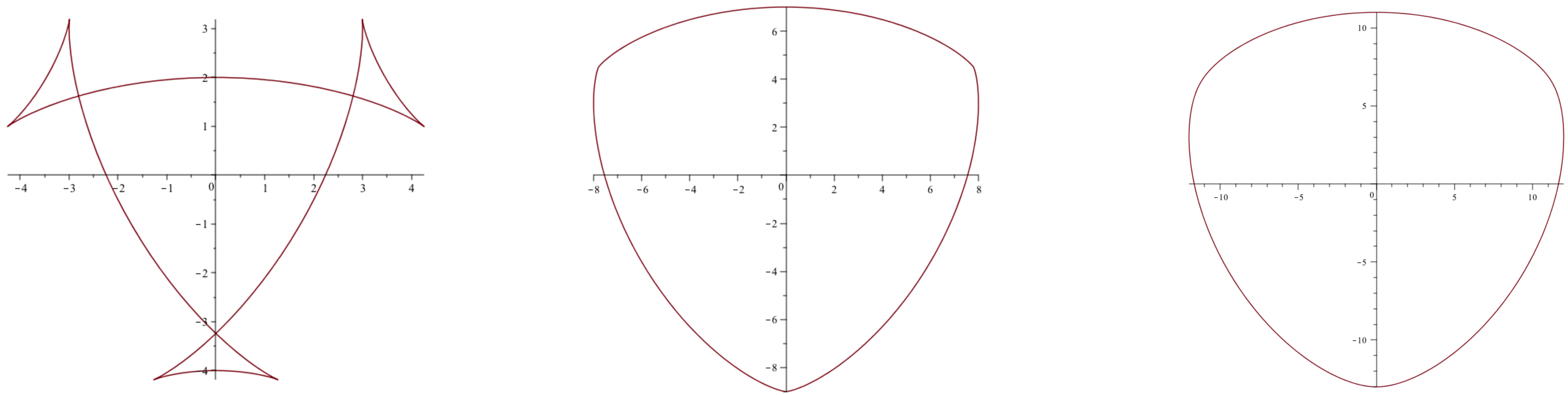

(a) $D = 6$

(b) $D = 16$

(c) $D = 24$

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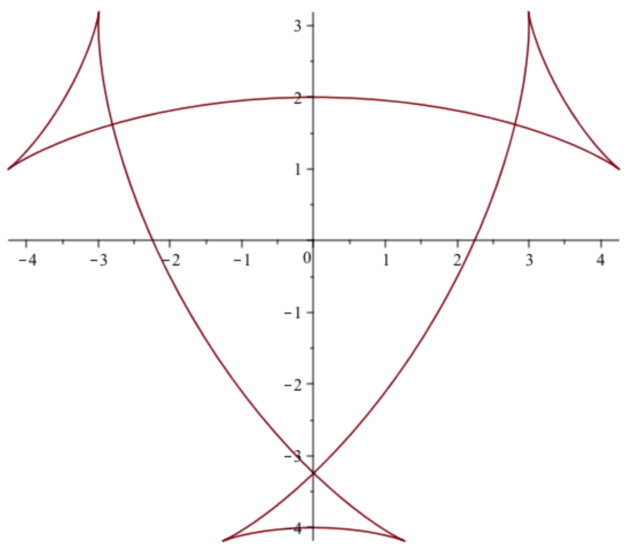

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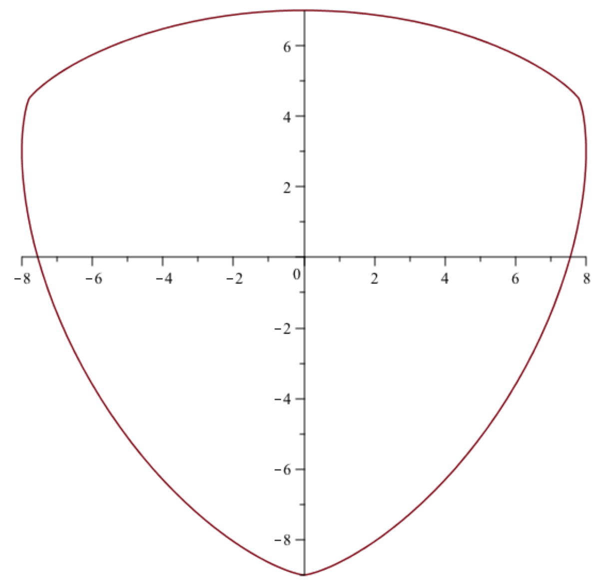

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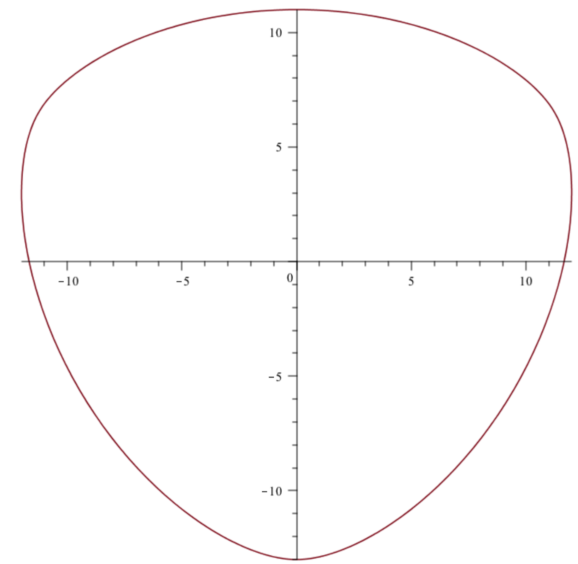

(c) $D = 24$

Convexity: $p(\theta) + p''(\theta) \geq 0$. This is related to the curvature of the curve. (Above, $D = 16$ is critical.)

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Theorem. Let $\mathcal{C}$ be any (convex) CCW of diameter D. Its circumference must be equal to πD. (This is a calculus exercise that you can do!)

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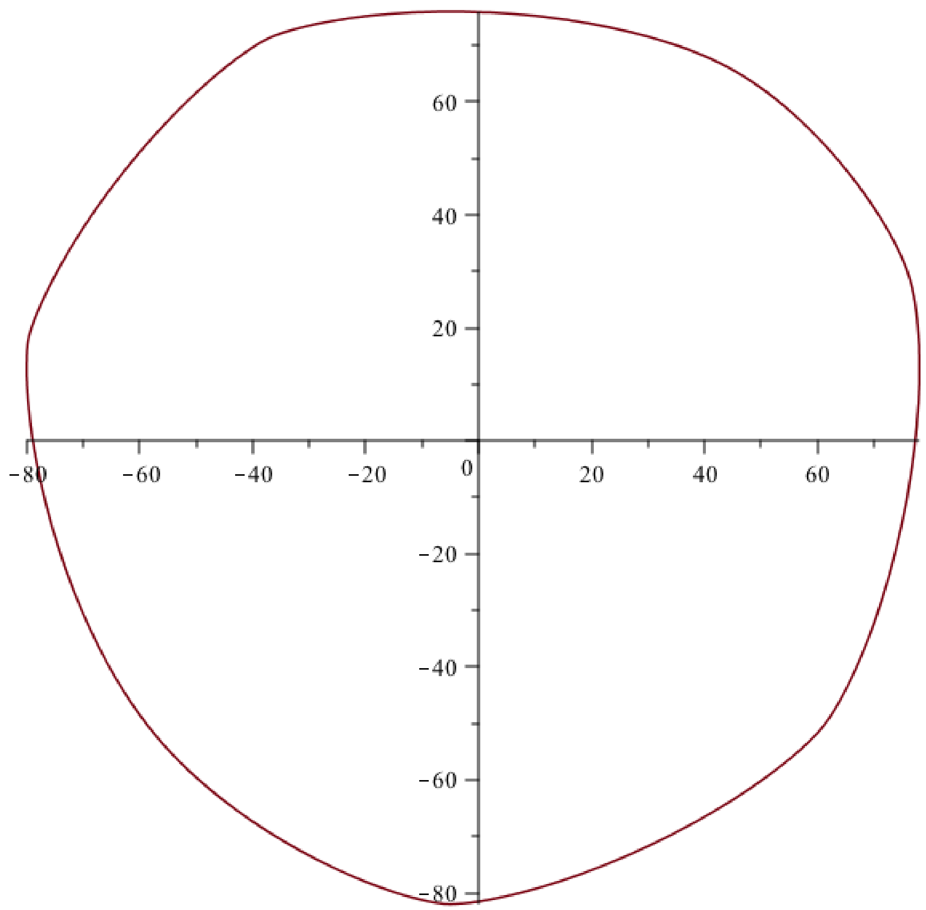
Another picture.



$$p(\theta) = 79 + 2\cos 3\theta - \sin 5\theta + \cos 7\theta$$

L. Variation — Equi-inscribable Curves (EIC)

Another view of CCW. A CCW of width D is a closed convex curve that can freely rotate between two parallel lines of distance D and touching both lines all the time.

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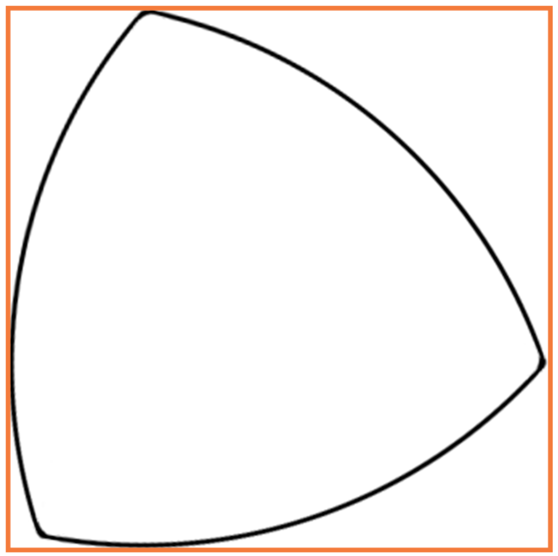
Note. Any CCW is a equi-inscribed in a square or a rhombus. Example:



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Q: Which polygons can admit noncircular curves that are equi-inscribed in them?

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Idea. Suppose that α, β are two outer angles of a triangle $\mathcal{T}$, with

$$0 < \alpha, \beta < \pi < \alpha + \beta.$$

A closed convex curve with support function $p(\theta)$ is equi-inscribed in a triangle similar to $\mathcal{T}$ if and only if

$$W_{\alpha,\beta}(\theta) := \sin(2\pi - \alpha - \beta)p(\theta) + \sin(\alpha)p(\theta + \beta) + \sin(\beta)p(\theta - \alpha)$$

is constant in θ.

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Note. As $\alpha, \beta \to \pi/2$, we have

$$W_{\alpha,\beta}(\theta) \to p\left(\theta + \frac{\pi}{2}\right) + p\left(\theta - \frac{\pi}{2}\right),$$

which is just the width function shifted in θ by $\pi/2$.

L. Variation — Equi-inscribable Curves (EIC)

Example. $\alpha = \beta = 2\pi/3$, $p(\theta) = 3 + \cos 2\theta$. In this case, $W_{\alpha,\beta}(\theta) = 9\sqrt{3}/2$.

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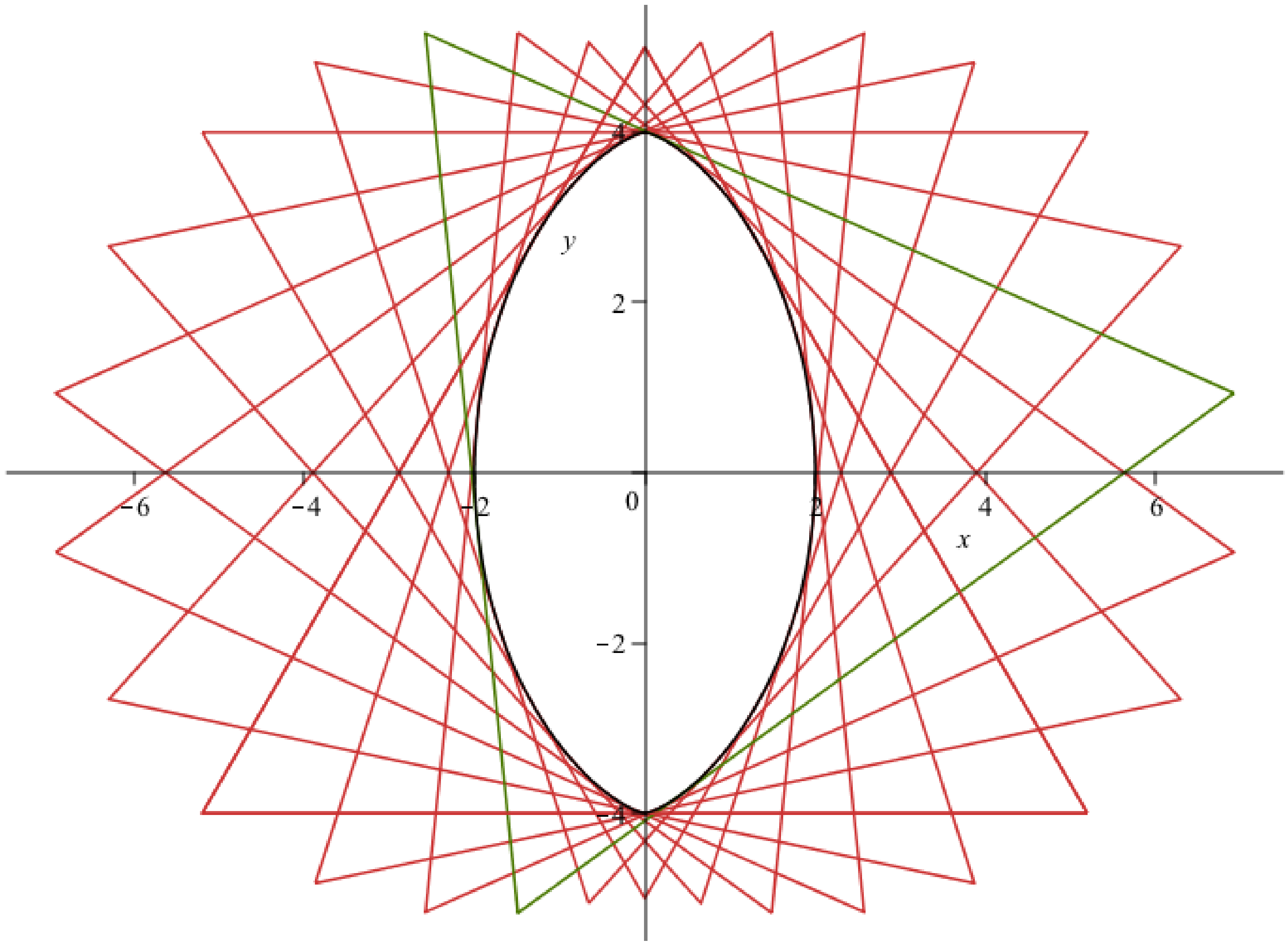

M. Beyond

Problem 1. Characterize curves that are equi-inscribable in quadrilaterals, pantagons, etc.

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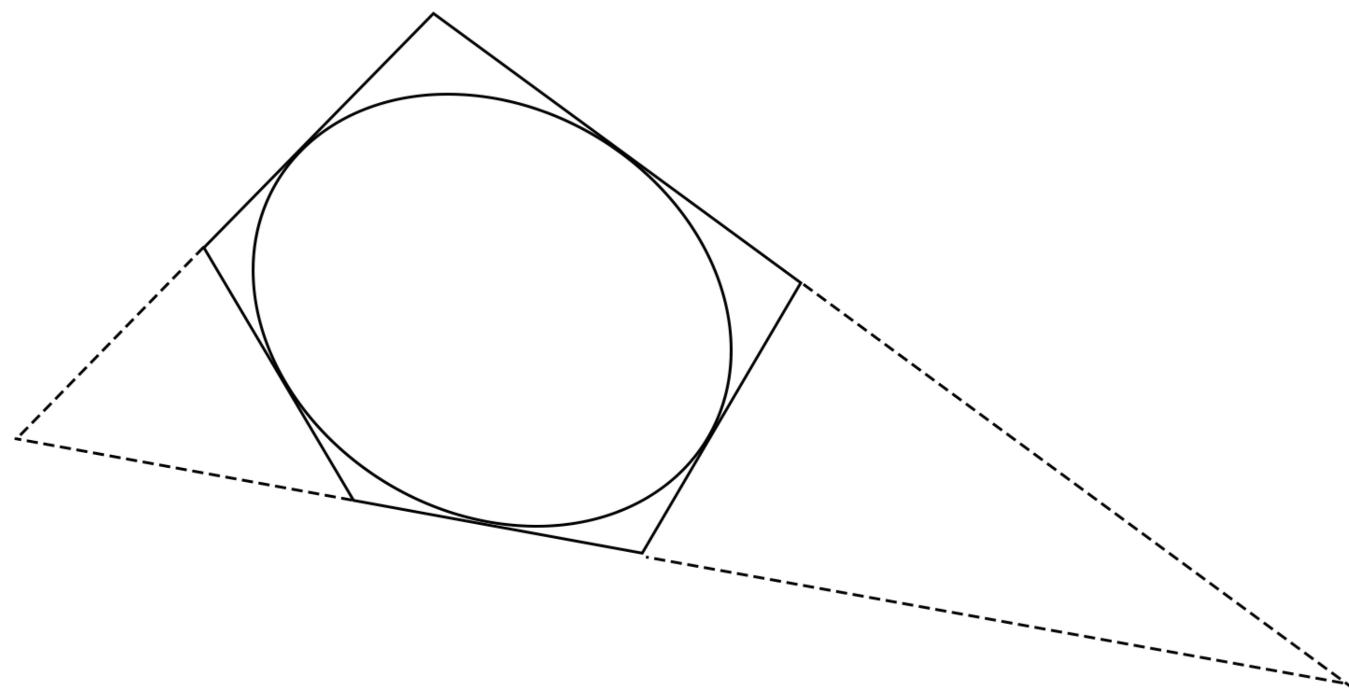

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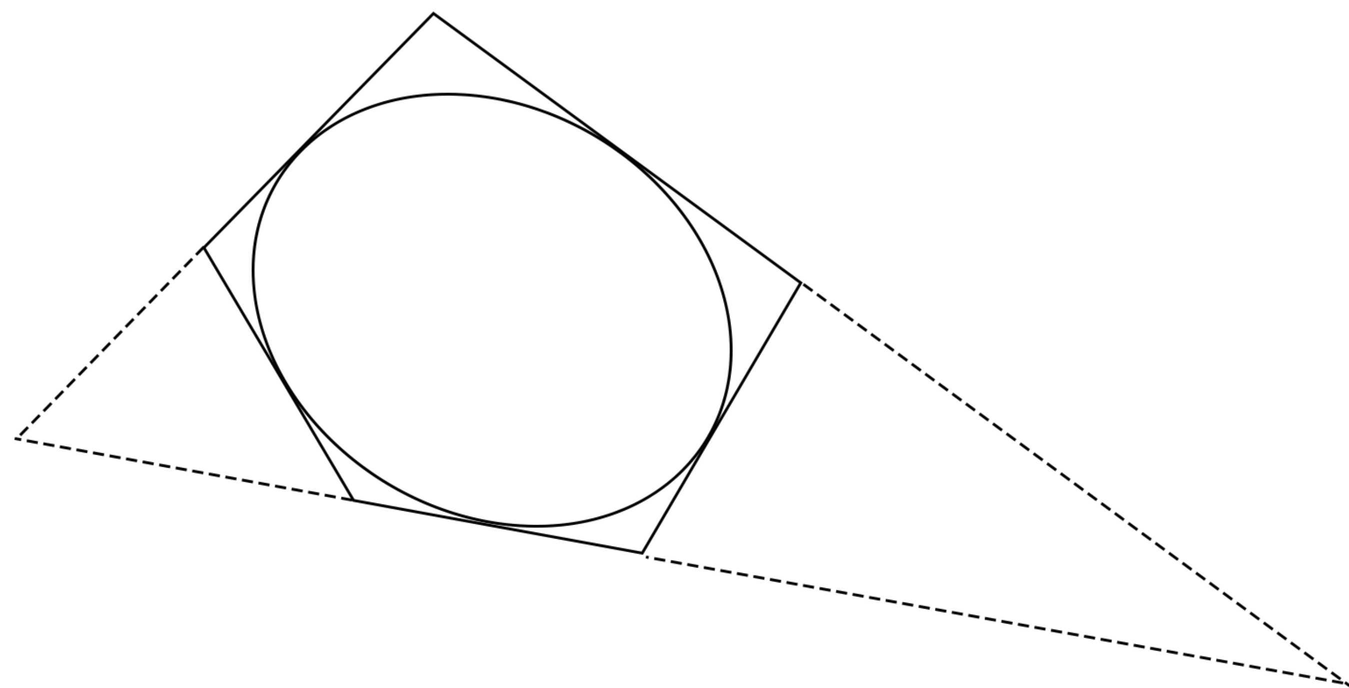

Q: What about sufficient conditions?

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Problem 2. From 2D to 3D — What about solids of constant width?

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Let $\mathcal{S} \subset \mathbb{R}^3$ be a closed, convex surface. Let $\mathbf{u}$ be the outer unit normal at a point $\mathbf{x} \in \mathcal{S}$. One can define a support function analogously by

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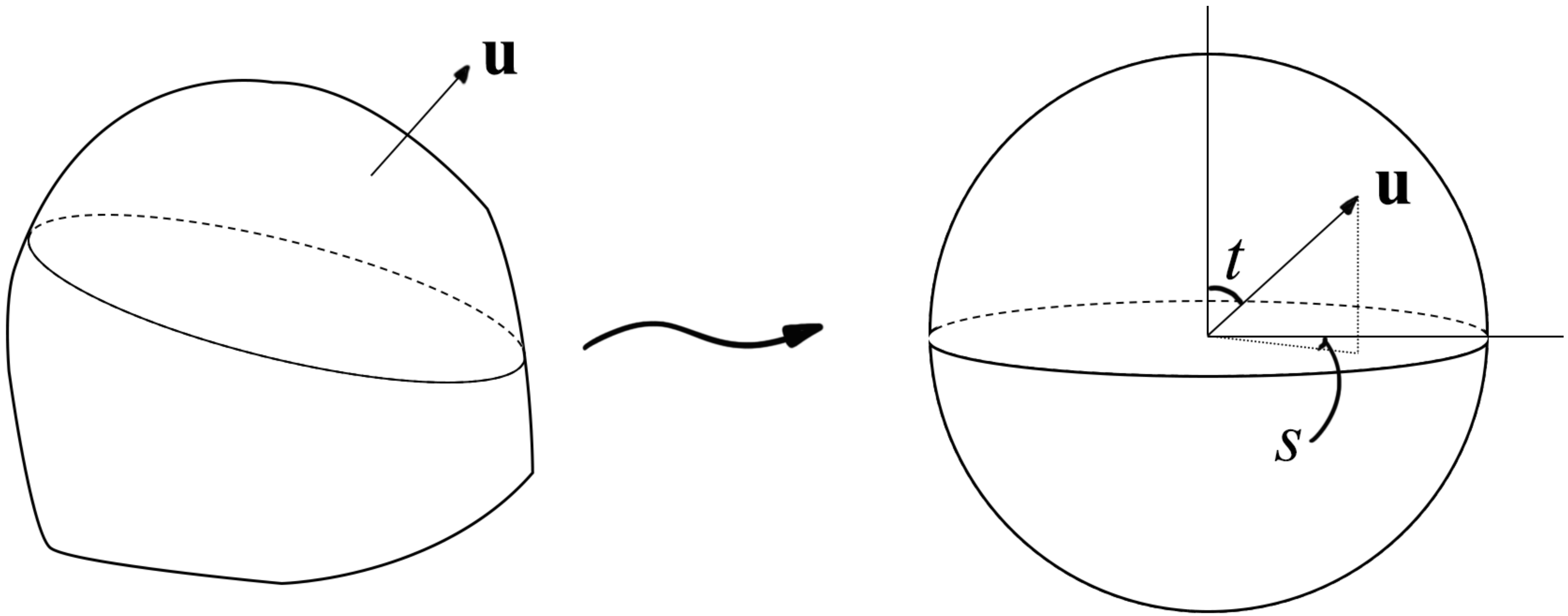

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$W = \mathrm{const.}$ is again a condition on the Fourier expansion of $p(s,t)$ (extended to be doubly periodic):

$$D = p(s,t) + p(s+\pi, \pi - t).$$

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Example. $p(\mathbf{u}) = 5/4 + u_1 u_2 u_3$. Clearly $p(\mathbf{u}) + p(-\mathbf{u}) = 5/2$, a constant. The surface has constant width and looks like:

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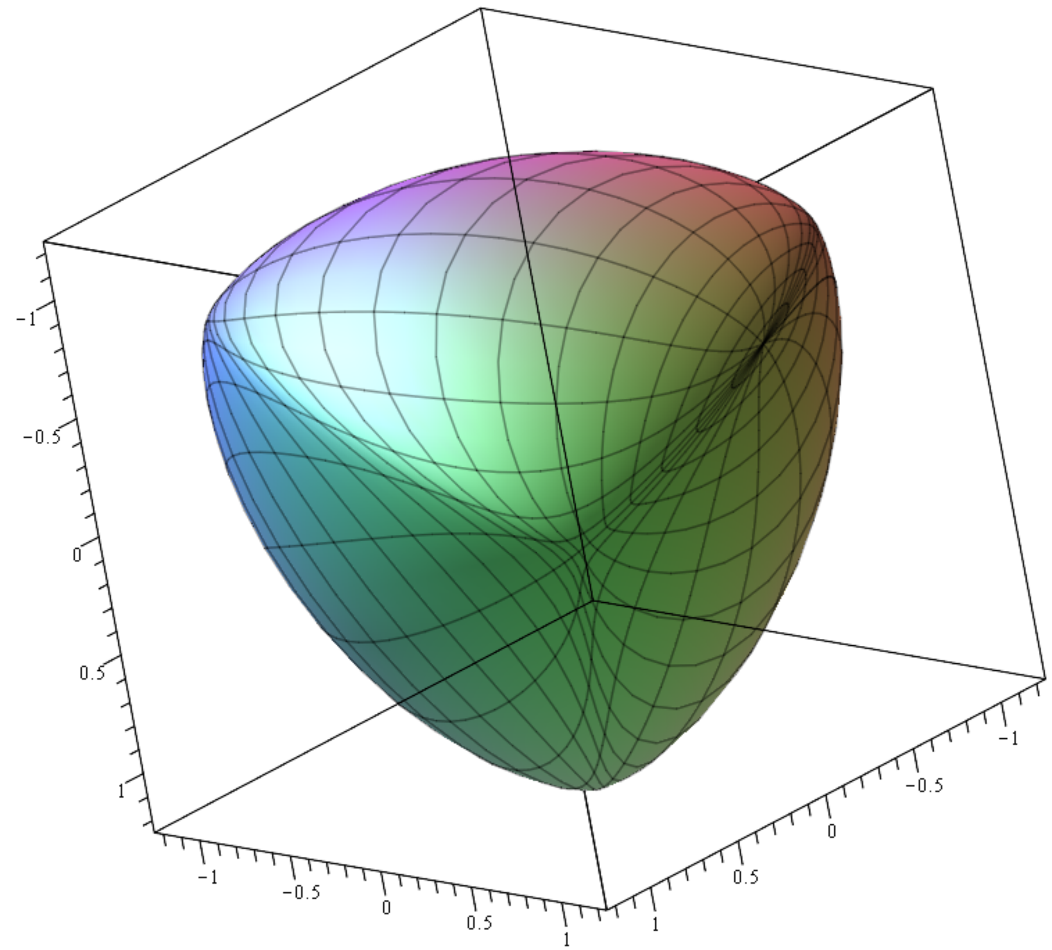

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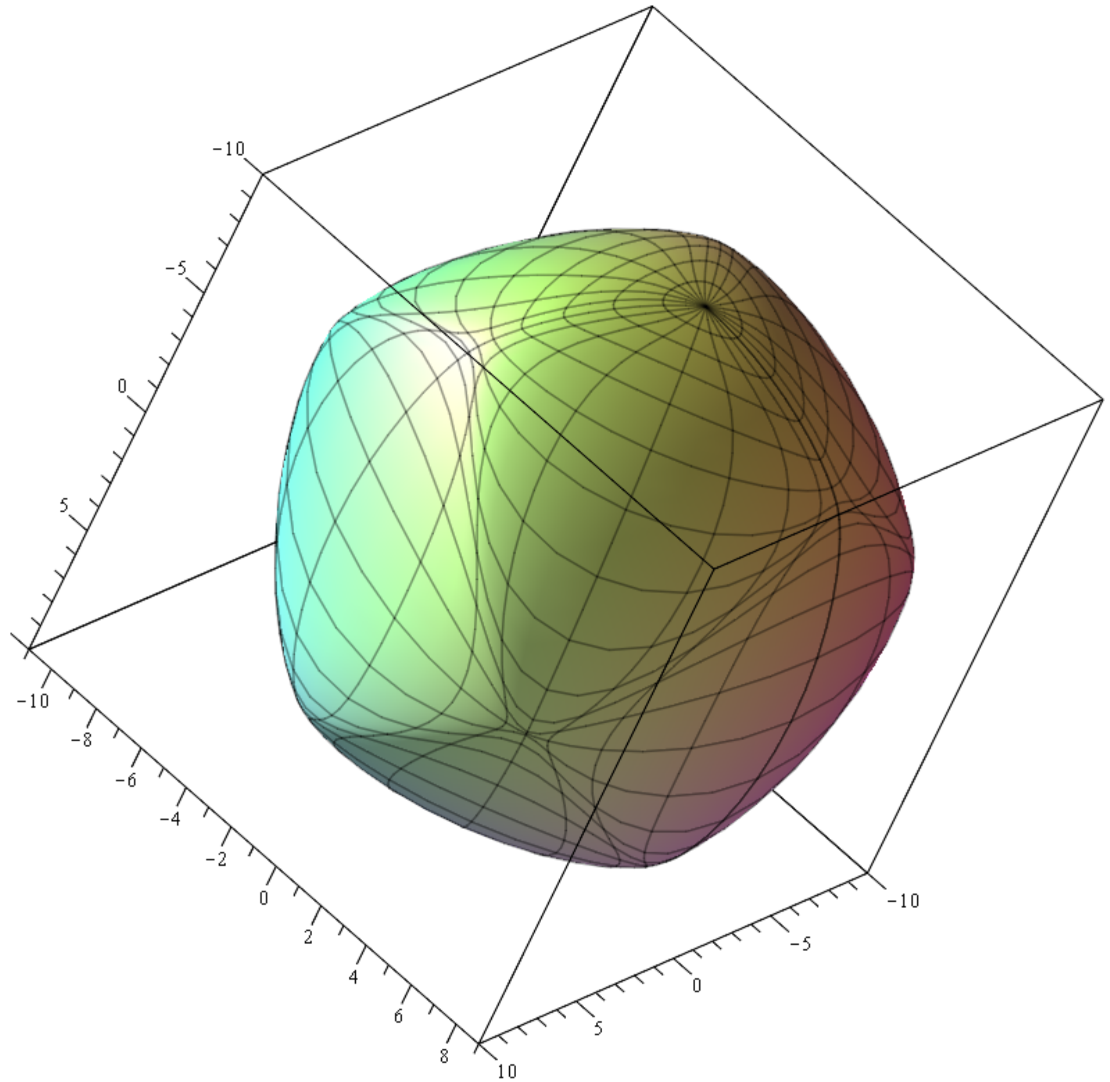

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Q: What else can we say about solids/surfaces of constant width? What about closed convex surfaces that are equi-inscribable in convex polyhedra (about which very little is known)?

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College Geometry Project (1965-71)

`https://archive.org/details/CollegeGeometry/Curves+of+Constant+Width.mkv`


Thank you!