DARBOUX INTEGRABILITY OF HYPERBOLIC SYSTEMS AND LIE'S THEOREM

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At the start of Chapter VII, Volume II of *Leçons sur L'Intégration des Équations aux Dérivées Partielle du Second Ordre*, E. Goursat wrote:

> *Pendant de longues anneées, après la publication des Mémoires d'Ampère (1814-1819), il n'a été ajouté rien d'essentiel à la théorie qu'il avait développée...L'année 1870 marque une date importante dans l'histoire de cette théorie; c'est en effet en mars 1870 que fut présenté à l'Académie des Sciences un remarquable Mémoire de M.Darboux, où se trouvent des vues profondes et originales...* (For long years after the publication of Memoirs of Ampère, nothing essential has been added to the theory he had developed...The year 1870 marks an important time in history of this theory; indeed, it is in March, 1870 that a remarkable Memoir by Darboux was presented to the Academy of Sciences, where profound and original views were obtained...)

The "profound" view of Darboux, as Goursat puts it, is a method of integration which enables one to solve certain types of hyperbolic PDEs with ODE techniques alone. With the more recent theory of exterior differential systems (EDS), this method has an geometric interpretation and that solutions can be found by integrating ODEs is almost immediate once the geometric picture is established. To explain this, I follow the approach taken by the more recent [BGH] and define *hyperbolic exterior differential systems of class s* and their *characteristic systems*; then, with a theorem which shows existence of certain adapted coframings, I define what's called *Darboux integrability at level k* and show by another theorem how(and why) Darboux's method works. These will be the topics for first part of the note.

A natural question that follows the Darboux's theorem is: *Which hyperbolic systems are Darboux integrable*? A moment's thought suggests that perhaps a more refined way of formulating this question is by asking: *Which hyperbolic systems of class s are Darboux integrable at level k*? Depending on the value of the two parameters s and k, the previous question can be either almost trivial to answer, or quite difficult with any theory available. For instance, in the case that class $s = 0$, Darboux integrability at level $k = 0$ is trivial; Darboux integrability at level $k = 1$ for non-degenerate systems is classified as having only two types up to local equivalence (see [BGH]). In the case that $s = 1$, it is an immediate consequence of the G-structure equations that the only Monge-Ampère systems that are Darboux integrable at level $k = 0$ are those equivalent to the classical wave equation $z_{xy} = 0$. By contrast, to my best knowledge, a complete classification of Monge-Ampère systems that are Darboux integrable at level $k = 1$ remains unknown to this day.

In light of the difficulty of classifying Darboux integrable systems as the parameters (s, k) grow large, the following Theorem of S.Lie appears to me as remarkable:

Theorem(S.Lie) *The only f-Gordon equations*

$$z_{xy} = f(z)$$

that are Darboux integrable at any level are locally equivalent to either the wave equation $z_{xy} = 0$ or the Liouville's equation $z_{xy} = e^z$.

This theorem will be explained in the second part and is the main purpose of this note. The proof of Lie's theorem I present follows almost verbatim the arguments in E.Goursat work.

1. Hyperbolic EDS, Characteristic systems and Darboux Integrability

Definition. *An exterior differential system $(M^{s+4}, \mathcal{I})$ is said to be* ***hyperbolic of class s*** *if there exists a coframing $(\theta_1, ..., \theta_s, \omega^1, \omega^2, \omega^3, \omega^4)$ on M so that*

$$\mathcal{I} = \{\theta_1, \theta_2, ..., \theta_s, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\}_{\text{alg}}.$$

Such coframings are said to be ***0-adapted*** *to the differential system $(M, \mathcal{I})$.*

Let $(M, \mathcal{I})$ be a hyperbolic EDS of of class s with a local 0-adapted coframing $(\theta_1, ..., \theta_s, \omega^1, ..., \omega^4)$ as defined above; and suppose that $E = \langle v \rangle$ is a 1-dimensional integral element of $(M, \mathcal{I})$ based at $p \in M$. The polar

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equations for $H(E)$ are clearly given by the vanishing of the 1-forms:

$$\theta_1, \theta_2, ..., \theta_s, v \lrcorner (\omega^1 \wedge \omega^2), v \lrcorner (\omega^3 \wedge \omega^4).$$

Thus, $H(E)$ has exactly dimension 2 (that is, E admits a *unique* extension to a 2-dimensional integrabl element) if and only if these 1-forms correspond to a system of linear equations of rank $s+2$ on T_pM. Equivalently speaking, E is a *characteristic* integral element if and only if either

$$(i) \quad \theta_1(v) = \theta_2(v) = ... = \theta_s(v) = \omega^1(v) = \omega^2(v) = 0,$$

or

$$(ii) \quad \theta_1(v) = \theta_2(v) = ... = \theta_s(v) = \omega^3(v) = \omega^4(v) = 0.$$

This motivates the

Definition. *Let $(M, \mathcal{I})$ be a hyperbolic EDS of class s, for which $(\theta_1, ..., \theta_s, \omega^1, ..., \omega^4)$ is a 0-adpated coframing. The **0-th characteristic systems** of $(M, \mathcal{I})$ are defined as the pair of Pfaffian systems:*

$$\Xi_{10} = \{\theta_1, ..., \theta_s, \omega^1, \omega^2\},$$
$$\Xi_{01} = \{\theta_1, ..., \theta_s, \omega^3, \omega^4\}.$$

Since the characteristic integral elements are intrinsic to a differential system, it is immediate that Ξ_{10} and Ξ_{01}, up to a switch of roles, are independent from the choice of 0-adapted coframings. Furthermore, the geometry of the characteristic variety Ξ of $(M, \mathcal{I})$ and the space of the first prolongation $M^{(1)}$ can be learned from a sequence of observations, as follows.

1. By the characteristic equations, the characteristic integral elements within T_pM are precisely all the 1-dimensional subspaces of the two vector spaces

$$V_{10} = \langle \theta_0, ..., \theta_s, \omega^1, \omega^2 \rangle_p^{\perp}, \quad V_{01} = \langle \theta_0, ..., \theta_s, \omega^3, \omega^4 \rangle_p^{\perp}.$$

Since $V_{10} \cap V_{01} = \{0\}$, it follows that the characteristic variety Ξ of $(M, \mathcal{I})$ is an $(\mathbb{RP}^1 \amalg \mathbb{RP}^1)$-bundle over M.

2. Let $v \in V_{10}, w \in V_{01}$ be nonzero vectors, it is clear that the plane $\langle v, w \rangle$ is an integral element of $(M, \mathcal{I})$. In other words, any projective line in $\mathbb{RP}^3$ intersecting both $\mathbb{RP}^1$-components in the fiber Ξ_p is a 2-dimensional integral element of the hyperbolic system.

3. Let $E_2 \subset T_pM$ be a 2-dimensional integral element. It is easy to see that, restricting to E_2, the 1-forms ω^1, ω^2 are linearly dependent, so are ω^3 and ω^4. Moreover, $\langle \omega^1, \omega^2, \omega^3, \omega^4 \rangle$ restricts to E_2 to have rank 2, thus one may assume that $\omega^1|_{E_2}, \omega^3|_{E_2}$ are independent. As a result, the equations $\omega^1 = 0$ and $\omega^3 = 0$ each define a 1-dimensional subspace of E_2 which is characteristic. Therefore, we have that the projective line in $\mathbb{RP}^3 \cong \mathbb{P}(T_pM)$ determined by any 2-dimensional integral element must intersect each of $\mathbb{P}(V_{10})$ and $\mathbb{P}(V_{01})$ at a point.

4. One learns immediately from the previous two observations that the space of the first prolongation $M^{(1)}$ is a $(\mathbb{RP}^1 \times \mathbb{RP}^1)$-bundle over M.

As for the differential ideal $\mathcal{I}^{(1)}$, if we restrict to an open set in $M^{(1)}$ consisting of 2-dimensional integral elements of $(M, \mathcal{I})$ at which the 1-forms ω^1, ω^3 are independent, then $M^{(1)}$ has fiber coordinates (h_1, h_3) with $E_2 \in M^{(1)}$ being defined by

$$E_2 = \langle \theta_1, ..., \theta_s, \omega^2 - h_1\omega^1, \omega^4 - h_3\omega^3 \rangle^{\perp}.$$

In this setting, locally, $\mathcal{I}^{(1)}$ is simply the Pfaffian system

$$\mathcal{I}^{(1)} = \{\theta_1, ..., \theta_s, \omega^2 - h_1\omega^1, \omega^4 - h_3\omega^3\}_{\text{diff}},$$

where the 1-forms are, of course, the pull-backs of respective forms on M. As a result, there exist functions T^i defined locally on $M^{(1)}$ such that

$$d\omega^i \equiv T^i\omega^1 \wedge \omega^3 \qquad \text{mod } \theta_1, ..., \theta_s, \omega^2 - h_1\omega^1, \omega^4 - h_3\omega^3.$$

A simple calculation leads to

$$\left.\begin{array}{l} d\theta_a \equiv 0 \\ d(\omega^2 - h_1\omega^1) \equiv -(dh_1 + (h_1T^1 - T^2)\omega^3) \wedge \omega^1 \\ d(\omega^4 - h_3\omega^3) \equiv -(dh_3 + (h_3T^3 - T^4)\omega^1) \wedge \omega^3 \end{array}\right\} \quad \text{mod } \theta_1, ..., \theta_s, \omega^2 - h_1\omega^1, \omega^4 - h_3\omega^3.$$

Hence, by introducing the 1-forms

$$\theta_{10} = \omega^2 - h_1\omega^1, \quad \theta_{01} = \omega^4 - h_3\omega^3, \quad \pi_{20} = dh_1 + (h_1T^1 - T^2)\omega^3, \quad \pi_{02} = dh_3 + (h_3T^3 - T^4)\omega^1,$$

we see that $(M^{(1)}, \mathcal{I}^{(1)})$ is a hyperbolic system of class $s+2$ with a local 0-adapted coframing:

$$(\theta_1, ..., \theta_s, \theta_{10}, \theta_{01}, \pi_{20}, \omega^1, \pi_{02}, \omega^3).$$

Needless to say, the k-th prolongation $(M^{(k)}, \mathcal{I}^{(k)})$ is hyperbolic of class $s+2k$ and 0-adapted coframings can be constructed iteratively from one on $(M, \mathcal{I})$. In fact, 0-adapated coframings on the k-th prolongation can be refined so as to satisfy certain structure equations. In particular, the following proposition is proven in [BGH].

Proposition. *Let $(M, \mathcal{I})$ be a hyperbolic EDS of class s. On the k-th prolongation $(M^{(k)}, \mathcal{I}^{(k)})$, a local coframing*

$$(\theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0}, \theta_{01}, ..., \theta_{0k}, \pi_{k+1,0}, \omega_{10}, \pi_{0,k+1}, \omega_{01})$$

exists satisfying

$$\mathcal{I}^{(k)} = \{\theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0}, \theta_{01}, ..., \theta_{0k}, \pi_{k+1,0} \wedge \omega_{10}, \pi_{0,k+1} \wedge \omega_{01}\}_{\text{alg}}$$

and

$$\begin{aligned}
d\theta_{k0} &\equiv -\pi_{k+1,0} \wedge \omega_{10} + T_{k0}\omega_{01} \wedge \theta_{01} && \text{mod } \theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0},\\
d\theta_{0k} &\equiv -\pi_{0,k+1} \wedge \omega_{01} + T_{0k}\omega_{10} \wedge \theta_{10} && \text{mod } \theta_1, ..., \theta_s, \theta_{01}, ..., \theta_{0k},\\
d\theta_{j0} &\equiv -\theta_{j+1,0} \wedge \omega_{10} + T_{j0}\omega_{01} \wedge \theta_{01} && \text{mod } \theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{j0},\\
d\theta_{0j} &\equiv -\theta_{0,j+1} \wedge \omega_{01} + T_{0j}\omega_{10} \wedge \theta_{10} && \text{mod } \theta_1, ..., \theta_s, \theta_{01}, ..., \theta_{0j},\\
d\theta_a &\equiv P_a\theta_{10} \wedge \omega_{10} + Q_a\theta_{01} \wedge \omega_{01} && \text{mod } \theta_1, ..., \theta_s,
\end{aligned}$$

*where $a = 1, ..., s$, and $j = 1, ..., k-1$. Such coframings are called **1-adapted** to the system $(M^{(k)}, \mathcal{I}^{(k)})$.*

The proof of this proposition is by induction and constructive in nature. Though not mentioned explicitely in the paper [BGH], one can insert a parellel argument along with the inductive steps which leads to the fact that the pair of (partial) flags

$$\begin{aligned}
\mathcal{F}_{10}^{(k)} =& (\langle\theta_1, ..., \theta_s\rangle \subset \langle\theta_1, ..., \theta_s, \theta_{10}, \omega_{10}\rangle \subset ... \subset \langle\theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0}, \omega_{10}\rangle\\
&\subset \langle\theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0}, \pi_{k+1,0}, \omega_{10}\rangle \subset T_x^*M^{(k)}),\\
\mathcal{F}_{01}^{(k)} =& (\langle\theta_1, ..., \theta_s\rangle \subset \langle\theta_1, ..., \theta_s, \theta_{01}, \omega_{01}\rangle \subset ... \subset \langle\theta_1, ..., \theta_s, \theta_{01}, ..., \theta_{0k}, \omega_{01}\rangle\\
&\subset \langle\theta_1, ..., \theta_s, \theta_{01}, ..., \theta_{0k}, \pi_{0,k+1}, \omega_{01}\rangle \subset T_x^*M^{(k)})
\end{aligned}$$

are well-defined at any $x \in U$ for some open $U \subset M^{(k)}$. This observation motivates the definition of the k-th characteristic system associated to $(M, \mathcal{I})$.

Definition. *The **k-th characteristic systems** $\Xi_{10}^{(k)}, \Xi_{01}^{(k)}$ of $(M, \mathcal{I})$ are locally defined as the Pfaffian systems:*

$$\begin{aligned}
\Xi_{10}^{(k)} &= \{\theta_1, ..., \theta_s, \theta_{10}, ..., \theta_{k0}, \pi_{k+1,0}, \omega_{10}\}_{\text{diff}},\\
\Xi_{01}^{(k)} &= \{\theta_1, ..., \theta_s, \theta_{01}, ..., \theta_{0k}, \pi_{0,k+1}, \omega_{01}\}_{\text{diff}}.
\end{aligned}$$

I remark that on any integral surface of $(M^{(k)}, \mathcal{I}^{(k)})$, the characteristic systems $\Xi_{01}^{(k)}$ and $\Xi_{10}^{(k)}$ restrict to be rank-one Frobenius systems, whose integral curves endows the integral surface with two foliations by characteristic curves.

We are now prepared to define the Darboux integrability of a hyperbolic system.

Definition. *A hyperbolic system $(M, \mathcal{I})$ of class s is said to be **Darboux integrable at level k** if there exist rank-two Frobenius systems $\Delta_{10}^{(k)} \subset \Xi_{10}^{(k)}$ and $\Delta_{01}^{(k)} \subset \Xi_{01}^{(k)}$ such that the associated vector bundles satisfy the independence conditions*

$$\Delta_{10}^{(k)} \cap (I^{(k)})_1 = \Delta_{01}^{(k)} \cap (I^{(k)})_1 = \langle 0\rangle.$$

Since the k-th characteristic system $\Xi_{10}^{(k)}$ of $\mathcal{I}$ is properly contained in the 0-th characteristic system $\Xi_{10}(\mathcal{I}^{(k)})$ of the k-th prolongation of $\mathcal{I}$, it is clear that if $\mathcal{I}$ is Darboux integrable at level k, then $\mathcal{I}^{(k)}$ is Darboux integrable at level zero. This directs us to first considering the case when $(M, \mathcal{I})$ is a hyperbolic system of class s and is Darboux integrable at level zero. Let $(\theta_1, ..., \theta_s, \omega^1, \omega^2, \omega^3, \omega^4)$ be a 0-adapted coframing and

$$\begin{aligned}
\Xi_{10} &= \{\theta_1, ..., \theta_s, \omega^1, \omega^2\},\\
\Xi_{01} &= \{\theta_1, ..., \theta_s, \omega^3, \omega^4\}.
\end{aligned}$$

By the assumption, Ξ_{10} (resp. Ξ_{01}) has two independent local first integrals $X, Y : M \supset U \to \mathbb{R}$ (resp. $P, Q : M \supset U \to \mathbb{R}$) whose exterior differentials do not lie in $\mathcal{I}_1$. In particular,

$$\begin{aligned}
\Xi_{10} &= \{\theta_1, ..., \theta_s, dX, dY\},\\
\Xi_{01} &= \{\theta_1, ..., \theta_s, dP, dQ\}.
\end{aligned}$$

Let $\gamma : (-\epsilon, \epsilon) \to U \subset M$ be a non-characteristic initial curve for $(M, \mathcal{I})$. This means that (i) γ' is everywhere annihilated by $\theta_1, ..., \theta_s$; and (ii) $\langle dX, dY \rangle$ (resp. $\langle dP, dQ \rangle$) restricts to γ to be rank-one. As a result, by choosing a smaller ϵ if needed, $\gamma_{XY} : (-\epsilon, \epsilon) \to \mathbb{R}^2$ defined by

$$\gamma_{XY}(t) = (X(\gamma(t)), Y(\gamma(t)))$$

is an immersed curve in the XY-plane. One similarly defines γ_{PQ} as a curve in the PQ-plane. Thus,

$$(\gamma_{XY} \times \gamma_{PQ})(s, t) = (\gamma_{XY}(s), \gamma_{PQ}(t))$$

defines an immersed surface $\mathcal{S}$ in the $XYPQ$-space. I also point out that, since, on U, $dX \wedge dY \wedge dP \wedge dQ \neq 0$, the map

$$\pi := (X, Y, P, Q) : U \to \mathbb{R}^4$$

is a local submersion. Moreover, π maps the curve γ to the diagonal of the immersed $\mathcal{S}$. Hence, $\pi^{-1}(\mathcal{S}) \subset U$ is a codimension-two submanifold of M which contains the initial curve γ and on which $dX \wedge dY = dP \wedge dQ = 0$. That is, the differential system $\mathcal{I}$ restricts to $\pi^{-1}(\mathcal{S})$ to be an integrable Pfaffian system of rank s and $\pi^{-1}(\mathcal{S})$ is foliated by integral surfaces of $\mathcal{I}$. In particular, $\gamma \subset \pi^{-1}(\mathcal{S})$ lies in a unique such integral surface. Noting that a Frobenius system can be integrated by ODE techniques, essentially we have proven the

Theorem. *If $(M, \mathcal{I})$ is a Darboux integrable hyperbolic system, then (locally) given any initial integral curve γ, there exists a unique integral surface $\mathcal{S}$ of $(M, \mathcal{I})$ containing γ; and $\mathcal{S}$ can be found by integrating a Frobenius system, i.e., by solving ordinary differential equations.*

As a trivial example, the wave equation in the plane $z_{xy} = 0$ can be established as a hyperbolic system on the $xyzpq$-space $M = J^1(\mathbb{R}^2, \mathbb{R})$ with the differential ideal

$$\mathcal{I} = \{dz - pdx - qdy, dp \wedge dx, dq \wedge dy\}.$$

It is obvious that the characteristic systems admit $\{dp, dx\}$, $\{dq, dy\}$ as rank-two integrable subsystems. The surface $\mathcal{S}$ determined by an non-characteristic initial curve γ generally take the form $\mathcal{S} : (s, t) \to (x(s), p(s), y(t), q(t))$. The space $\pi^{-1}(\mathcal{S})$ is parametrized by z, s, t and $\mathcal{I}$ restrict to be generated by the single 1-form

$$dz - p(s)x'(s)ds - q(t)y'(t)dt,$$

which is clearly closed. As a result, z takes the form $z(s, t) = F(s) + G(t)$. If $x'(s), y'(t) \neq 0$, then we can write $s = s(x), t = t(y)$ and obtain the familiar $z(x, y) = F(s(x)) + G(t(y)) = \bar{F}(x) + \bar{G}(y)$.

A less trivial example is the Liouville system

$$\begin{cases} u_y = e^v \\ v_x = e^u \end{cases}$$

This corresponds to a hyperbolic EDS on the $xyuv$-space $M \subset J^0(\mathbb{R}^2, \mathbb{R}^2)$ with the differential ideal

$$\mathcal{I} = \{(du - e^v dy) \wedge dx, (dv - e^u dx) \wedge dy\}.$$

Clearly, $(M, \mathcal{I})$ is not Darboux integrable at level zero, but it turns out that it is Darboux integrable at level 1. Technically, checking this amounts to computing the characteristic systems of the first prolongation $(M^{(1)}, \mathcal{I}^{(1)})$ and their derived systems. Once the rank-two Frobenius subsystems $\Delta_{10}^{(1)}$ and $\Delta_{01}^{(1)}$ are obtained, manipulating the 1-forms so that they become closed leads to the four first integrals X, Y, P, Q, expressed as functions of x, y, u, v, p, q, where p, q are new coordinate functions introduced on $M^{(1)}$. Given an non-characteristic initial curve γ, the corresponding surface $\mathcal{S}$ is then given in coordinates $S : (s, t) \to (X(s), Y(s), P(t), Q(t))$. Now $\pi^{-1}(\mathcal{S})$ submerses onto $\mathcal{S}$, thus, restricting to $\pi^{-1}(S)$, four of the functions among x, y, u, v, p, q, say u, v, p, q, can be locally solved for in terms of the rest of the variables and s, t; then the differential ideal pulls back to the $xyst$-space to be a rank-two Frobenius system, whose first integrals can be found without much difficulty.

What's notable about the Liouville system is that its solutions satisfy

$$u_{xy} = v_{xy} = e^{u+v}.$$

Hence, $u + v$ satisfies the Liouville equation

$$z_{xy} = 2e^z.$$

Thus, in certain sense, we have seen that both the planar wave equation $z_{xy} = 0$ and the Liouville equation $z_{xy} = e^z$ are Darboux integrable at some level. These two types of equations are special cases of the more general f-Gordon equations $z_{xy} = f(z)$. By a theorem of Sophus Lie in 1880, these are the *only* f-Gordon equations are Darboux integrable at *any* level. I will discuss Lie's proof in the next section.

2. Lie's Theorem on Darboux Integrability

Lie's proof of the theorem stated in the introduction of this note can be seen to be carried out in three steps.

1. Establish the hyperbolic exterior differential systems $(M,\mathcal{I})$ corresponding to the f-Gordon equation $z_{xy} = f(z)$, compute the k-th characteristic systems $\Xi_{10}^{(k)}$, $\Xi_{01}^{(k)}$, and notice that $dx \in \Xi_{10}^{(k)}$ and $dy \in \Xi_{01}^{(k)}$. Observing symmetry, one only needs to work in one of the characteristic systems, say, $\Xi_{10}^{(k)}$.
2. Show that any first integral ψ for $\Xi_{10}^{(k)}$ must be the first integral of a rank-two distribution $\mathcal{D}_{10}$ which is a sub-distribution of the completely integrable $\langle \partial_x, \partial_y, \partial_z, \partial_{p_1}, ..., \partial_{p_{k+1}} \rangle$ on $M^{(k)}$.
3. Show that if $(f')^2 - f''f \neq 0$, then the infinite derivation $\mathcal{D}_{10}^{\infty}$ is a corank-one subbundle of $\langle \partial_x, \partial_y, \partial_z, \partial_{p_1}, ..., \partial_{p_{k+1}} \rangle$, hence x is the only first integral of $\Xi_{10}^{(k)}$. (For a distribution $\mathcal{D}$, the k-the derivation $\mathcal{D}^k$ is defined inductively via $\mathcal{D}^0 = \mathcal{D}$ and $\mathcal{D}^{k+1} = \mathcal{D}^k \cup [\mathcal{D}^k, \mathcal{D}^k]$. The infinite derivation $\mathcal{D}^{\infty}$ is by definition $\bigcup_{i=0}^{\infty} \mathcal{D}^i$.)

Step 1. To start with, note that $z_{xy} = f(z)$ can be established as a hyperbolic EDS on $M = J^1(\mathbb{R}^2, \mathbb{R})$ with coordinates (x, y, z, p_1, q_1). The 1-forms

$$\begin{aligned}\theta &= dz - p_1 dx - q_1 dy,\\ \pi_{10} &= dp_1 - f(z)dy,\\ \omega_{10} &= dx,\\ \pi_{01} &= dq_1 - f(z)dx,\\ \omega_{01} &= dy\end{aligned}$$

form a 0-adapted coframing on M such that the differential ideal $\mathcal{I}$ can be written as

$$\mathcal{I} = \{\theta, \pi_{10} \wedge \omega_{10}, \pi_{01} \wedge \omega_{01}\}_{\text{alg}}.$$

To compute the first prolongation $(M^{(1)}, \mathcal{I}^{(1)})$, we introduce extra coordinates p_2, q_2 on $M^{(1)}$ such that a two-dimensional integral element based at $(x, y, z, p_1, q_1) \in M^{(1)}$ is determined by the vanishing of the 1-forms

$$\theta, \pi_{10} - p_2 dx, \pi_{01} - q_2 dy.$$

If we let

$$\theta_{10} = \pi_{10} - p_2 dx, \quad \theta_{01} = \pi_{01} - q_2 dy,$$

then the prolonged differential ideal is simply the Pfaffian system

$$\mathcal{I}^{(1)} = \{\theta, \theta_{10}, \theta_{01}\}_{\text{diff}}.$$

Exterior differentiating θ_{10} and θ_{01} gives

$$\begin{aligned}d\theta_{10} &\equiv -(dp_2 - f'(z)p_1 dy) \wedge dx && \mod \theta,\\ d\theta_{01} &\equiv -(dq_2 - f'(z)q_1 dx) \wedge dy && \mod \theta.\end{aligned}$$

So we could put

$$\begin{aligned}\pi_{20} &= dp_2 - f'(z)p_1 dy,\\ \pi_{02} &= dq_2 - f'(z)q_1 dx,\end{aligned}$$

and the coframing $\{\theta, \theta_{10}, \theta_{01}, \pi_{20}, dx, \pi_{02}, dy\}$ becomes automatically 1-adapted for $(M^{(1)}, \mathcal{I}^{(1)})$. Soon one realizes the symmetry between computing the systems $\Xi_{10}^{(k)}$ and $\Xi_{01}^{(k)}$, so I shall from now on present only the results leading to $\Xi_{10}^{(k)}$ for brevity.

Naturally, on $M^{(2)}$, p_3 is introduced and

$$\theta_{20} = \pi_{20} - p_3 dx,$$

is one of the generators for $\mathcal{I}^{(2)}$. Differentiation gives

$$d\theta_{20} \equiv -(dp_3 - (f'(z)p_2 + f''(z)(p_1)^2)dy) \wedge dx \qquad \mod \theta, \theta_{10},$$

hence π_{30} can be introduced as

$$\pi_{30} = dp_3 - (f'(z)p_2 + f''(z)(p_1)^2)dy,$$

such that

$$d\theta_{20} \equiv -\pi_{30} \wedge dx \qquad \mod \theta, \theta_{10}.$$

If we continue, on $M^{(3)}$, p_4 is introduced and

$$\begin{aligned}
\theta_{30} &= \pi_{30} - p_4 dx,\\
d\theta_{30} &\equiv -(dp_4 - (f'(z)p_3 + f'''(z)(p_1)^3 + 3f''(z)p_1p_2)dy)\wedge dx, \qquad \mod \theta, \theta_{10}, \theta_{20}.
\end{aligned}$$

Hence, π_{40} can be given as

$$\pi_{40} = dp_4 - (f'(z)p_3 + f'''(z)(p_1)^3 + 3f''(z)p_1p_2)dy.$$

Having in mind what is going on simultaneously for the $(0i)$-subindexed forms, I point out that $(\theta, \theta_{10}, ..., \theta_{03}, \pi_{40}, dx, \pi_{04}, dy)$ is automatically a 1-adapted coframing for $(M^{(3)}, \mathcal{I}^{(3)})$ and that the 3$^{\text{rd}}$ characteristic systems of $(M, \mathcal{I})$ are simply

$$\begin{aligned}
\Xi^{(3)}_{10} &= \{\theta, \theta_{10}, \theta_{20}, \theta_{30}, \pi_{40}, dx\},\\
\Xi^{(3)}_{01} &= \{\theta, \theta_{01}, \theta_{02}, \theta_{03}, \pi_{04}, dy\}.
\end{aligned}$$

It is possible now to examine some patterns for the coframes introduced on $M^{(k)}$, which I summarize as the following

Lemma 1. *A 1-adapted coframing $\{\theta, \theta_{10}, ..., \theta_{k0}, \theta_{01}, ..., \theta_{0k}, \pi_{k+1,0}, dx, \pi_{0,k+1}, dy\}$ for $(M^{(k)}, \mathcal{I}^{(k)})$ can be constructed progressively such that*

$$\begin{aligned}
d\theta_{j0} &\equiv -\theta_{j+1,0}\wedge dx \qquad \mod \theta, \theta_{10}, \theta_{20}, ..., \theta_{j-1,0}, \quad (j = 1, ..., k-1)\\
d\theta_{k0} &\equiv -\pi_{k+1,0}\wedge dx \qquad \mod \theta, \theta_{10}, ..., \theta_{k-1,0},\\
\pi_{k+1,0} &= dp_{k+1} - (f'(z)p_k + Q_{k-1})dy,\\
\theta_{i0} &= dp_i - (f'(z)p_{i-1} + Q_{i-2})dy - p_{i+1}dx, \quad (i = 1, ..., k)
\end{aligned}$$

and similarly for $\theta_{0j}, \theta_{0k}, \pi_{0,k+1}$ and $\theta_{0,k+1}$. Here Q_i are polynomials in $p_1, ..., p_i$ whose coefficients are functions of z. For non-positive i, we put p_i, Q_i to be identically zero.

As a result of the lemma, the k-th characteristic system $\Xi^{(k)}_{10}$ of $(M, \mathcal{I})$ is

$$\Xi^{(k)}_{10} = \{\theta, \theta_{10}, ..., \theta_{k0}, \pi_{k+1,0}, dx\},$$

and an obvious first integral of $\Xi^{(k)}_{10}$ is the function x.

To summarize Lie's further observations, I name polynomials in $p_1, ..., p_i$ with coefficients being functions of z as ***z-polynomials of level i***. In addition, we make the

Definition. *Let the **weight** of a nonzero z-monomial $P = g(z)p_1^{i_1}...p_r^{i_r}$ be defined as $w(P) = i_1 + 2i_2 + ... + ri_r$, and define the **weight of a z-polynomial** to be the maximum among the weights of its nonzero terms. For completeness, let $w(0) = -\infty$.*

Then it is easy to prove by induction the following

Lemma 2. *The z-polynomials Q_i $(i \geq 1)$ constructed in Lemma 1 satisfy*

$$w(Q_i) \leq i + 1.$$

Proofs of Lemma 1 and Lemma 2 will be given in Appendix II.

Step 2. Suppose that ψ is a first integral for $\Xi^{(k)}_{10}$. By the expressions of the 1-forms $\theta, \theta_{10}, ..., \theta_{k0}, \pi_{k+1,0}$ given in Lemma 1, ψ must be independent of $q_1, ..., q_{k+1}$. As a result,

$$d\psi \equiv \left(\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial z}q_1 + \frac{\partial\psi}{\partial p_1}f(z) + \frac{\partial\psi}{\partial p_2}f'(z)p_1 + \sum_{i=3}^{k+1}\frac{\partial\psi}{\partial p_i}(f'(z)p_{i-1} + Q_{i-2})\right)dy \qquad \mod \Xi^{(k)}_{10}.$$

Of course, the coefficient of dy in the equation above must vanish. Since ψ is a function of $x, y, z, p_1, ..., p_{k+1}$, and that $\frac{\partial\psi}{\partial z}q_1$ is the only term involving some q_i in the coefficient, we have obtained the two equations

$$\begin{cases} A(\psi) = 0,\\ B(\psi) = 0,\end{cases}$$

where A, B are vector fields on $M^{(k)}$ given by

$$A = \frac{\partial}{\partial z},$$
$$B = \frac{\partial}{\partial y} + f(z)\frac{\partial}{\partial p_1} + f'(z)p_1\frac{\partial}{\partial p_2} + \sum_{i=3}^{k+1}(f'(z)p_{i-1} + Q_{i-2})\frac{\partial}{\partial p_i}.$$

Clearly, $\{A, B\}$ pointwisely span a rank-distribution $\mathcal{D}_{10}$ on $M^{(k)}$ and ψ is a first integral of $\mathcal{D}_{10}$.

Step 3. The rest of the proof amounts to taking Lie brackets in clever ways so as to lead to the desired rank argument about $\mathcal{D}_{10}^{\infty}$.

First, we compute

$$[A, B] = f'(z)\frac{\partial}{\partial p_1} + f''(z)p_1\frac{\partial}{\partial p_2} + \sum_{i=3}^{k+1}\left(f''(z)p_{i-1} + \frac{\partial Q_{i-2}}{\partial z}\right)\frac{\partial}{\partial p_i}.$$

Comparing the expressions of $[A, B]$ and B, Lie had the idea of considering the combinations

$$f'(z)B - f(z)[A, B], \quad f''(z)B - f'(z)[A, B],$$

and noted that, if $(f')^2 - f''f$ is non-vanishing, then one can scale to obtain

$$M = \frac{1}{(f')^2 - f''f}(f'(z)B - f(z)[A, B]) = \frac{f'}{(f')^2 - f''f}\frac{\partial}{\partial y} + p_1\frac{\partial}{\partial p_2} + \sum_{i=3}^{k+1}(p_{i-1} + S_{i-2})\frac{\partial}{\partial p_i},$$
$$N = \frac{1}{(f')^2 - f''f}(f''(z)B - f'(z)[A, B]) = -\frac{f''}{(f')^2 - f''f}\frac{\partial}{\partial y} + \frac{\partial}{\partial p_1} + \sum_{i=3}^{k+1}T_{i-2}\frac{\partial}{\partial p_i},$$

where S_i, T_i are z-polynomials of level i with weight $\leq i+1$.

Now, define let $N^{(0)} = N$ and define $N^{(k)}$ $(k \geq 1)$ by the formula

$$N^{(k)} = [M, N^{(k-1)}].$$

We have

$$N^{(1)} = \frac{\partial}{\partial p_2} + \sum_{i=3}^{k+1}V_{i-2}^{(1)}\frac{\partial}{\partial p_i},$$

where, for each $i \in \{3, 4, ..., k+1\}$,

$$V_{i-2}^{(1)} = \frac{\partial T_{i-2}}{\partial p_2}p_1 + \frac{\partial S_{i-2}}{\partial p_1} + \sum_{j=3}^{k+1}(p_{j-1} + S_{j-2})\frac{\partial T_{i-2}}{\partial p_j} + \sum_{j=3}^{k+1}\left(\frac{\partial p_{i-1}}{\partial p_j} + \frac{\partial S_{i-2}}{\partial p_j}T_{j-2}\right)$$

is easily seen to be a z-polynomial of level $i-2$ with weight $\leq i-2$. Indeed, more can be proven:

Lemma 3. *For $j = 1, ..., k+1$ there exist z-polynomials $V_i^{(j)}$ of level i and of weight $\leq i$ such that*

$$N^{(j)} = (-1)^{j+1}\frac{\partial}{\partial p_{j+1}} + \sum_{i=j+2}^{k+1}V_{i-j-1}^{(j)}\frac{\partial}{\partial p_i}.$$

Proof of Lemma 3 is included in Appendix II. Letting $j = k$ in the lemma, we have

$$N^{(k)} = (-1)^{k+1}\frac{\partial}{\partial p_{k+1}}.$$

Tracing back, it is easy to see that

$$\langle A, B, M, N, N^{(1)}, ..., N^{(k)}\rangle = \langle \partial_y, \partial_z, \partial_{p_1}, ..., \partial_{p_{k+1}}\rangle.$$

Consequently, the differential of ψ must be a multiple of dx.

Finally, I remark that the the only functions $f(z)$ satisfying $f''f = (f')^2$ are of the form $f(z) = \lambda e^{\mu z}$, where λ, μ are constants. This completes the proof of Lie's theorem.

3. Further Questions

1. The notion of Darboux integrability at level k appears by definition more restricted than the notion of the k-th prolongation of a system being Darboux integrable. The latter is essentially what's being used for proving that Darboux's method works. Is there any hyperbolic system whose k-th prolongation is Darboux integrable but is itself not Darboux integrable at level k?
2. In the same spirit as the previous question, is there any f-Gordon equation not locally equivalent to either $z_{xy} = 0$ or $z_{xy} = e^z$ whose k-th prolongation (for some k) is Darboux integrable?

4. Appendix I. Conventions on Notations

I use the round bracket (...) to express a coframing on a manifold. Differential or algebraic ideals generated by certain differential forms are put in curled brackets such as $\{...\}_{\text{diff}}$ or $\{...\}_{\text{alg}}$. Sometimes, the plain $\{...\}$ is used to mean $\{...\}_{\text{diff}}$. The angled bracket $\langle ... \rangle$, with input(s) being either sections of a vector bundle or generators of a vector space, means the linear-generation of a vector bundle or a vector space.

A differential ideal will be denoted using script letters such as $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. The set of k-forms in a differential ideal $\mathcal{I}$ is denoted as $\mathcal{I}_k$. The vector bundle associated to $\mathcal{I}_k$ is denoted as I_k. The k-th prolongation of $\mathcal{I}$ has the notation $\mathcal{I}^{(k)}$. To avoid confusion, I use $(I^{(k)})_1$ rather than $I_1^{(k)}$ to denote the vector bundle of 1-forms in $\mathcal{I}^{(k)}$, since the latter may be mistakenly understood as the k-th prolongation of the Pfaffian system generated by I_1. For a Pfaffian system such as $\Xi_{10}^{(k)}$, the same notation can be used to represent the vector bundle generated by all the 1-forms in $\Xi_{10}^{(k)}$, or the differential ideal whose elements are local sections of vector bundles. The k-th derived system of a Pfaffian system I, which is also Pfaffian, has the notation $I^{\langle k \rangle}$. If the derived (partial) flag ... $\subset I^{\langle 2 \rangle} \subset I^{\langle 1 \rangle}$ stabilizes at some $I^{\langle s \rangle}$, then $I^{\langle s \rangle}$ is naturally the maximal Frobenius subsystem of I, and is given the notation $I^{\langle \infty \rangle}$. To conclude by an example, the third derived system of the system algebraically generated by the 1-forms in the second prolongation of $\mathcal{I}$ is denoted as $(I^{(2)})_1^{\langle 3 \rangle}$.

All problems in the range of this exposition are *local*, so when I write, for instance, "condition $\mathcal{C}$ holds on M", what I really mean is "condition $\mathcal{C}$ holds on some open subset $U \subset M$". Sometimes, when multiple such open subsets are concerned, one might worry about empty overlaps. In such cases, one could first ask whether or not the open subset U can be chosen to be dense in M.

5. Appendix II. Proofs of Lemmas

Proof of Lemma 1 and Lemma 2. By the previous calculation, it is clear that this lemma holds for $k = 1, 2, 3$. Therefore, it suffices to carry out the inductive step from k to $k + 1$. By 1-adaptedness of the coframing

$$(\theta, \theta_{10}, ..., \theta_{k0}, \theta_{01}, ..., \theta_{0k}, \pi_{k+1,0}, dx, \pi_{0,k+1}, dy),$$

one can introduce new coordinates p_{k+2}, q_{k+2} such that the Pfaffian system $\mathcal{I}^{(k+1)}$ is given by

$$\mathcal{I}^{(k+1)} = \{\theta, \theta_{10}, ..., \theta_{k0}, \theta_{k+1,0}, \theta_{01}, ..., \theta_{0k}, \theta_{0,k+1}\},$$

where

$$\theta_{k+1,0} = \pi_{k+1,0} - p_{k+2} dx, \quad \theta_{0,k+1} = \pi_{0,k+1} - q_{k+1} dy.$$

Clearly, on $M^{(k+1)}$,

$$d\theta_{j0} \equiv -\theta_{j+1,0} \wedge dx \qquad \text{mod } \theta, \theta_{10}, ..., \theta_{j-1,0}, \quad (j = 1, ..., k).$$

Furthermore,

$$\begin{aligned} d\theta_{k+1,0} &= d\pi_{k+1,0} - dp_{k+2} \wedge dx \\ &= -\left(f''(z)dz + f'(z)dp_k + \frac{\partial Q_{k-1}}{\partial z} dz + \sum_{i=1}^{k-1} \frac{\partial Q_{k-1}}{\partial p_i} dp_i \right) \wedge dy - dp_{k+2} \wedge dx \\ &\equiv -\left(f''(z)p_1 + f'(z)p_{k+1} + \frac{\partial Q_{k-1}}{\partial z} p_1 + \sum_{i=1}^{k-1} \frac{\partial Q_{k-1}}{\partial p_i} p_{i+1} \right) dx \wedge dy - dp_{k+2} \wedge dx \\ &\qquad\qquad \text{mod } \theta, \theta_{10}, ..., \theta_{k0}. \end{aligned}$$

If we let

$$Q_k = f''(z)p_1 + \frac{\partial Q_{k-1}}{\partial z} p_1 + \sum_{i=1}^{k-1} \frac{\partial Q_{k-1}}{\partial p_i} p_{i+1},$$

and use the inductive assumption that for each $i \in \{1, 2, ..., k - 2\}$, Q_i is a z-polynomials of level i and has weight $\leq i + 1$, then it can be easily seen that Q_k is a z-polynomial of level k and with weight $\leq k + 1$.

As a result, define

$$\pi_{k+2,0} = dp_{k+2} - (f'(z)p_{k+1} + Q_k)dy,$$

and we have

$$d\theta_{k+1,0} \equiv -\pi_{k+2,0} \wedge dx \qquad \text{mod } \theta, \theta_{10}, ..., \theta_{k0}.$$

Since the argument for the the other characteristic system is completely analogous to the above, we have proven Lemma 1 and 2.

Proof of Lemma 3. Note that if R_1, R_2 are two z-polynomials, then the weights satisfy

$$w(R_1R_2) \leq w(R_1)w(R_2).$$

Here we did not specify the levels of R_1, R_2, since, in a sense, the weights give upper-bounds for the minimal levels. With this note, the proof of Lemma 3 reduces to pure calculations. Suppse that for some $1 \leq j < k$ we have

$$N^{(j)} = (-1)^{j+1}\frac{\partial}{\partial p_{j+1}} + \sum_{i=j+2}^{k+1} V^{(j)}_{i-j-1}\frac{\partial}{\partial p_i},$$

where each $V^{(j)}_\ell$ is a z-polynomial of level i and with weight $\leq i$. We compute, by letting $g(z) = f'/((f')^2 - f''f)$,

$$\begin{aligned}
[M, N^{(j)}] &= \left[g(z)\frac{\partial}{\partial y} + p_1\frac{\partial}{\partial p_2} + \sum_{i=3}^{k+1}(p_{i-1} + S_{i-2})\frac{\partial}{\partial p_i} \quad , \quad (-1)^{j+1}\frac{\partial}{\partial p_{j+1}} + \sum_{i=j+2}^{k+1} V^{(j)}_{i-j-1}\frac{\partial}{\partial p_i} \right] \\
&= (-1)^{j+2}\sum_{i=3}^{k+1}\frac{\partial}{\partial p_{j+1}}(p_{i-1} + S_{i-2})\frac{\partial}{\partial p_i} + \sum_{\ell=j+2}^{k+1}\sum_{i=3}^{k+1}(p_{i-1} + S_{i-2})\frac{\partial V^{(j)}_{\ell-j-1}}{\partial p_i}\frac{\partial}{\partial p_\ell} \\
&\quad - \sum_{\ell=3}^{k+1}\sum_{i=j+2}^{k+1} V^{(j)}_{i-j-1}\frac{\partial(p_{\ell-1} + S_{\ell-2})}{\partial p_i}\frac{\partial}{\partial p_\ell}.
\end{aligned}$$

In the first term of the result above, the summation splits into

$$\sum_{i=3}^{k+1}\frac{\partial p_{i-1}}{\partial p_{j+1}}\frac{\partial}{\partial p_i} \quad \text{and} \quad \sum_{i=3}^{k+1}\frac{\partial S_{i-2}}{\partial p_{j+1}}\frac{\partial}{\partial p_i}.$$

The former is simply

$$\frac{\partial}{\partial p_{j+2}},$$

and the summation in the latter actually starts with $i = j+3$. The weight of $\frac{\partial S_{i-2}}{\partial p_{j+1}}$ is clearly $\leq i - j - 2$.

For the second term, the outer summation can start from $\ell = j+3$, since $\frac{\partial V^{(j)}_1}{\partial p_i} = 0$ for $i \geq 3$. The coefficient of $\frac{\partial}{\partial p_\ell}$ are easily seen to have weight $\leq \ell - j - 2$.

In the third term, the outer summation can start from $\ell = j+3$, since $\frac{\partial(p_{\ell-1}+S_{\ell-2})}{\partial p_i}$ is identically zero for $\ell \leq j+3$ and $i \geq j+2$. Evidently, the weight of the coefficient of $\frac{\partial}{\partial \ell}$ is $\leq i - j - 2$.

To sum up, $[M, N^{(j)}]$ can be put in the form

$$[M, N^{(j)}] = (-1)^{j+2}\frac{\partial}{\partial p_{j+2}} + \sum_{i=j+3}^{k+1} V^{(j+1)}_{i-j-2}\frac{\partial}{\partial p_i},$$

where V^i is a z-polynomial with weight $\leq i$ and hence is of level i. This completes the proof.

6. Bibliography

[BGH] R.Bryant, P.Griffiths, L.Hsu, *Hyperbolic Exterior Differential Systems and their Conservations Laws*, Selecta Mathematica, New Series 1(1) (1995): 21-112.

[Go] E.Goursat, *Leçons sur L'Intégration des Équations aux Dérivées Partielle du Second Ordre*, Tome 2, Hermann, Paris (1898).