CALCULUS OF VARIATIONS VIA EXTERIOR DIFFERENTIAL SYSTEMS

YUHAO HU

In this note, I only consider 1-dimensional variational problems (i.e., variation by curves) where the variation is smooth and compactly supported. By "a variation" I mean a map

$$\Gamma : (-\epsilon, \epsilon) \times (a, b) \to M,$$

where M is a smooth manifold and $\Gamma(0, \cdot)$ is the initial curve. By "compactly supported" I mean that $\Gamma(\cdot, t)$ is constant for all t outside a compact subset of (a, b). By "smooth" I mean Γ is smooth. From now on, the term "variation" will be used carrying these meanings unless specified otherwise.

1. Classical Formulation and the Euler-Lagrange System

Let $\gamma : (a, b) \to \mathbb{R}^n$ represent a smooth curve in $\mathbb{R}^n$. A classically well-known variational problem is searching for the extremals of the Lagrangian functional

$$\mathcal{L} = \int_a^b L(t, \gamma(t), \gamma'(t)) dt$$

under variations of γ. One immediately notices the difference between the space on which L is defined and the space on which a variation takes place. An easy fix to this is by considering variations in the space $J^1(\mathbb{R}, \mathbb{R}^n) \cong \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(t, (q_i), (p_i))$, and restricting to curves along which the 1-forms $dq_i - p_i dt$ vanish and the 1-form dt non-vanish. It is then clear that these variations correspond uniquely to variations in $\mathbb{R}^n$. (The descriptive term "compactly supported" plays a role here, since, for example, the lifting to $J^1(\mathbb{R}, \mathbb{R}^n)$ of a variation in $\mathbb{R}^n$ with fixed end-points may not still have the end-points fixed.)

If we denote $J^1(\mathbb{R}, \mathbb{R}^n)$ as M, the vector subbundle of T^*M spanned by $dq_i - p_i dt$ ($i = 1, ..., n$) as I, and the 1-form $L dt$ (on M) as ϕ. The Lagrangian variational problem as above is then encoded in the triple (M, I, ϕ). To be precise, one searches for extremal curves with respect to the functional $\mathcal{L}(\gamma) = \int_\gamma \phi$ under (compactly supported) variations by integral curves of I. In the same manner, any triple (M, I, ϕ) in which M is a smooth manifold, $I \subset T^*M$ a vector subbundle, and $\phi \in \Omega^1(M)$ a smooth 1-form represents a variational problem.

In particular, suppose that $I = \{0\}$, and let $(s, t) \mapsto \Gamma(s, t)$ be a variation. Denote the vector field $\Gamma_*(\partial_s)$ as X (extended smoothly to a neighborhood of γ, if needed), its flow as Φ_s, and $\Gamma(s, \cdot)$ as γ_s. Thus, $\gamma_s = \Phi_s \circ \gamma$, and

$$\frac{d}{ds}\bigg|_{s=0} \int_{\gamma_s} \phi = \frac{d}{ds}\bigg|_{s=0} \int_{\Phi_s \circ \gamma} \phi = \frac{d}{ds}\bigg|_{s=0} \int_\gamma \Phi_s^* \phi = \int_\gamma \frac{d}{ds}\bigg|_{s=0} \Phi_s^* \phi = \int_\gamma \mathcal{L}_X \phi = \int_\gamma d(X \lrcorner \phi) + X \lrcorner d\phi.$$

The integral $\int_\gamma d(X \lrcorner \phi)$ vanishes by the compactly-supportedness assumption. Thus γ is an extremal if and only if

$$\int_\gamma X \lrcorner d\phi = \int_a^b d\phi(X, \gamma') dt = 0$$

for all compactly supported variational vector fields X. Since any compactly supported vector field X_γ along γ could extend to a variational vector field, a necessary and sufficient condition for γ to be extremal is

$$\gamma' \lrcorner d\phi \equiv 0.$$

Equivalently, letting $X_1, ..., X_n$ be local vector fields defined on $U \subset M$ which span $T_x M$ at each $x \in U$, then $\gamma \subset U$ is an extremal of (M, I, ϕ) if and only if γ is an integral curve of the differential system

$$\mathcal{C}(d\phi) = \{X_i \lrcorner d\phi\}_{i=1,...,n}.$$

This differential system is called the *Euler-Lagrange system* of $(M, \{0\}, \phi)$. In terms of exterior differential systems, $\mathcal{C}(d\phi)$ is the *Cartan system* of $d\phi$ and an integral curve γ of $\mathcal{C}(d\phi)$ is called a *characteristic* of $d\phi$.

Date: 11/02/2016.

2. The Griffiths Formalism

Given a variational problem (M, I, ϕ), one could define an associated variational problem $(Z, \{0\}, \zeta)$ where $Z = I + \phi \subset T^*M$ is an affine subbundle of T^*M and ζ is the canonical 1-form on Z. In coordinates, if locally I has rank s with basis sections $\theta^1, ..., \theta^s$, then a point $z \in Z$ can be written as

$$z = \phi_p + \lambda_i \theta_p^i,$$

and

$$\zeta_z = \pi^* \phi_p + \lambda_i \pi^* \theta_p^i,$$

where $\pi : Z \to M$ is the canonical submersion and $p = \pi(z)$.

Griffiths proposes studying the variational problem (M, I, ϕ) through $(Z, \{0\}, \zeta)$, in light of the fact that extremals of the latter satisfy the Euler-Lagrange system, which takes simple form. This would be a beautiful story as long as correspondence between extremals of these two variational problems can be established. Indeed, it is proven in [Bryant87] that

Any extremal of $(Z, \{0\}, \zeta)$ projects via π to be an extremal of (M, I, ϕ).

The proof is in fact quite simple. Since any extremal γ on Z must integrate the Euler-Lagrange system, we have

$$\gamma' \lrcorner \, d\zeta = 0,$$

where

$$d\zeta = d\pi^* \phi + d\lambda_i \wedge \pi^* \theta^i + \lambda_i d\pi^* \theta^i.$$

Modulo 1-forms that are semi-basic to π, it is easy to see that $\gamma' \lrcorner \, d\zeta$ is congruent to $-\theta^i(\pi_* \gamma') d\lambda_i$, which shows that $\pi \circ \gamma$ is an integral curve of I; also, it is easy to see that γ' cannot be a combination of ∂_{λ_i} alone, hence $\pi \circ \gamma$ is an immersed curve in M. Moreover, letting α_s be a variation of $\pi \circ \gamma$ by integral curves of I with $\alpha_0 = \pi \circ \gamma$, any lifting γ_s of α_s to Z with $\gamma_0 = \gamma$ is easily seen to satisfy

$$\int_{\gamma_s} \zeta = \int_{\alpha_s} \phi.$$

Clearly, $\alpha_0 = \pi \circ \gamma$ is an extremal since $\gamma_0 = \gamma$ is.

However, the converse question turns out to be subtle:

Which extremals of (M, I, ϕ) arise as projections of extremals of $(Z, \{0\}, \zeta)$?

The subtlety can be partially seen through the existence of rigid integral curves of certain (M, I, ϕ), where, by γ being "rigid" I mean any smooth compactly supported variation of γ by integral curves of I is only a reparametrization of γ itself. Such γ's can certainly be understood as extremals of (M, I, ϕ), but their liftings to $(Z, \{0\}, \zeta)$ need not satisfy the Euler-Lagrange system. (Intuitively, the rigid integral curves may have sufficient vector fields along them that are first order candidates for a variation, but a vector fields as such may not admit an extension to an actual variational vector field.)

As a sufficient condition, [Hsu92] obtains:

Any regular extremal of (M, I, ϕ) has a unique lifting to an extremal of $(Z, \{0\}, \zeta)$.

Of course, I need to explain what is meant by "regular". In [Hsu92], the definition of regularity is via the surjectivity of a holonomy map . To make sense of this, it requires a discussion of the variational equations (roughly, these are equations that give necessary conditions for a vector field along an integral curve γ of I to be associated to a variation by integral curves of I); and perhaps also a discussion about a consequence of the holonomy map's being surjective, which is, solutions to the variational equations are actual variational vector fields. Taking this approach would almost certainly digress too far away from my purpose of this note. Indeed, [Hsu92] shows that there is a characterization of *regularity* which is equivalent to its original definition. Here I shall take this equivalent characterization as the definition for simplicity:

*An integral curve γ of I is said to be **regular** if any lifting of γ to $Z_0 = I \subset T^*M$ to be a characteristic of $d\zeta_0$ intersects the zero section of Z_0; where ζ_0 is the canonical 1-form defined on Z_0.*

As a result of Hsu's theorem, regular extremals of (M, I, ϕ) can be studied by lifting regular integral curves of I to Z and examine whether or not the liftings integrate the Euler-Lagrange system of $(Z, \{0\}, \zeta)$. The power of this idea can be seen via several examples which I'll give next.

3. EXAMPLES

3.1. Geodesics on a Riemannian Surface.

Let (Σ, g) denote a Riemannian surface, $\mathscr{F}(\Sigma)$ its orthonormal frame bundle. On $\mathscr{F}(\Sigma)$, one can define three 1-forms ω^1, ω^2, ρ which satisfy the structure equations

$$d\omega^1 = \rho \wedge \omega^2,$$
$$d\omega^2 = -\rho \wedge \omega^1,$$
$$d\rho = -K\omega^1 \wedge \omega^2,$$

where K is a function which is constant on the fibers of the submersion $\mathscr{F}(\Sigma) \to \Sigma$, hence an invariant of the surface. Indeed, K is simply the well-known Gauss curvature of (Σ, g).

Given a smooth immersed curve γ in Σ, parametrized by arclength parameter s, there is a unique way to attach an orthonormal frame (e_1, e_2) along γ such that $\gamma' = e_1$ and the pair (e_1, e_2) agrees with the orientation of Σ. This lifting of γ to $\mathscr{F}(\Sigma)$, say $\hat{\gamma}$, satisfies

$$\hat{\gamma}^* \omega^1 = ds, \quad \hat{\gamma}^* \omega^2 = 0,$$

and there exists a function κ along $\hat{\gamma}$ such that

$$\hat{\gamma}^* \rho = \kappa \cdot \hat{\gamma}^* \omega^1;$$

this κ is known as the *geodesic curvature* of γ.

Now, consider the variational problem (M, I, ϕ), where $M = \mathscr{F}(\Sigma)$, $I = \{\omega^2\}$, and $\phi = \omega^1$. This is clearly a variational problem whose extremals are length-minimizing curves. Suppose that $\alpha : (a, b) \to M$ is an integral curve of I, we proceed to examine the condition for α to be regular.

On $Z_0 = I \subset T^*M$, the canonical 1-form ζ_0 can be written as (dropping the pull-back symbols for simplicity)

$$\zeta_0 = \lambda \omega^2,$$

hence

$$d\zeta_0 = d\lambda \wedge \omega^2 - \lambda \rho \wedge \omega^1.$$

The Cartan system $\mathcal{C}(d\zeta_0)$ is therefore generated by the 1-forms

$$d\lambda, \quad \omega^2, \quad \lambda \rho, \quad \lambda \omega^1.$$

Clearly, along any lifting of α, we have $\omega^1 \neq 0$; thus the only possibility for such a lifting to integrate $\mathcal{C}(d\zeta_0)$ is to lift to the zero section, i.e., $\lambda \equiv 0$. Therefore, by our definition, all integral curves of I in M are regular; hence extremals of (M, I, ϕ) and those of $(Z, \{0\}, \zeta)$ have one-to-one correspondence, according to Hsu's theorem.

On $(Z, \{0\}, \zeta)$, a point can be written as

$$z = \omega^1 + \lambda \omega^2.$$

With this notation, and dropping the pull-back symbols for simpicity, we have

$$\zeta = \omega^1 + \lambda \omega^2.$$

As a result,

$$d\zeta = \rho \wedge \omega^2 + d\lambda \wedge \omega^2 - \lambda \rho \wedge \omega^1.$$

The Euler-Lagrange system on Z is then generated by the 1-forms

$$\omega^2 - \lambda \omega^1, \quad \omega^2, \quad d\lambda + \rho, \quad \lambda \rho.$$

Since, along α, the pull-back of ω^2 vanishes; the same holds for any lifting of α. Hence, the vanishing of the 1-forms in the Euler-Lagrange system restricts on any lifting to be equivalent to the equations

$$\lambda = \kappa = 0.$$

This implies that, α has a lifting to Z to be an extremal (which is equivalent to α itself being an extremal) if and only if α has zero geodesic curvature, or, simply, α is a geodesic.

3.2. The Poincaré Problem.

Let (Σ, g) denote a Riemannian surface which is diffeomorphic to $\mathbb{S}^2$. Suppose that γ is a smooth simple closed curve on Σ which, according to its orientation, encloses a piece of surface $\Omega \subset M$ with $\partial\Omega = \gamma$. The Poincaré problem seeks such a γ with minimal length so that the total curvature

$$\int_\Omega K dA$$

of the enclosed surface is some prescribed constant K_0. According to the Gauss-Bonnet formula,

$$\int_\Omega K dA + \int_\gamma \kappa ds = 2\pi,$$

the integral constraint may be rewritten as

$$\int_\gamma \kappa ds = K_1 = 2\pi - K_0.$$

As in the previous example, this variational problem is best described using the frame bundle $\mathscr{F}(\Sigma)$. To deal with the integral constraint, we consider instead the space $\mathscr{F}(\Sigma) \times \mathbb{R}$ as M, where $\mathbb{R}$ has coordinate z, and restrict to variations by integral curves of $I = \{\omega^2, dz - \rho\}$. The reason of doing so is that, along an integral curve of ω^2, the 1-form ω^1 has the natural interpretation as ds where s is the arclength parameter of the corresponding curve in Σ; and ρ has the interpretation of being κds. A compactly supported variation α_s in the space $\mathscr{F}(M) \times \mathbb{R}$ will then necessarily preserve $\int_{\alpha_0} dz$, which is exactly $\int_\gamma \kappa ds$. To complete the setting, let $\phi = \omega^1$, so (M, I, ϕ) is a variational problem corresponding to the Poincaré problem.

To see which integral curves of I are regular, we simply note that the canonical 1-form on $Z_0 = I \subset T^*M$ can be written as

$$\zeta_0 = \lambda_1 \omega^2 + \lambda_2(dz - \rho),$$

where λ_1, λ_2 are coordinates on the fibers of $Z_0 \to M$. Exterior differentiation gives

$$d\zeta_0 = d\lambda_1 \wedge \omega^2 - \lambda_1 \rho \wedge \omega^1 + d\lambda_2 \wedge (dz - \rho) + \lambda_2 K \omega^1 \wedge \omega^2.$$

The Cartan system of $d\zeta_0$ is then generated by the 1-forms

$$\omega^2, \quad dz - \rho, \quad d\lambda_1 + \lambda_2 K \omega^1, \quad \lambda_1 \rho + \lambda_2 K \omega^2, \quad d\lambda_2, \quad \lambda_1 \omega^1.$$

Suppose that γ is an integral curve of I, then any lifting of γ to Z_0 must satisfy $\omega^2 = dz - \rho = 0$. Hence, the Cartan system restricted to a lifting of γ reduce to the equations

$$\lambda_1 = 0, \quad \lambda_2' = 0, \quad \lambda_2 K = 0,$$

where the derivatives are taken with respect to s defined by $ds = \gamma^* \omega^1$. It is clear that $(\lambda_1(s), \lambda_2(s))$ satisfying the equations above must vanish for some s if and only if $K|_\gamma$ is not identically zero. This gives the characterization for γ to be regular. In particular, if on Σ we have $K > 0$ everywhere (in other words, Σ is a convex surface), then all integral curves of I in M are regular.

To see which regular integral curves of I are extremals, note that the canonical 1-form on $Z = \omega^1 + I$ can be written as

$$\zeta = \omega^1 + \lambda_1 \omega^2 + \lambda_2(dz - \rho).$$

Thus,

$$d\zeta = \rho \wedge \omega^2 + d\lambda_1 \wedge \omega^2 - \lambda_1 \rho \wedge \omega^1 + d\lambda_2 \wedge (dz - \rho) + \lambda_2 K \omega^1 \wedge \omega^2.$$

The corresponding Cartan system is easily seen to be generated by

$$\omega^2, \quad dz - \rho, \quad \lambda_1 \rho + \lambda_2 K \omega^2, \quad d\lambda_1 + \rho + \lambda_2 K \omega^1, \quad d\lambda_2, \quad \omega^2 - \lambda_1 \omega^1.$$

Since γ is assumed to be an integral curve of I, the vanishing of these 1-forms on a lifting is equivalent to the equations

$$\lambda_1 = 0, \quad \lambda_2' = 0, \quad \kappa + \lambda_2 K = 0.$$

Hence, such a lifting exists if and only if, along γ, we have $\kappa = cK$ for some constant c. In other words, regular extremals in (M, I, ϕ) are characterized by the condition that the ratio between the geodesic and the Gauss curvatures along γ is a constant.

In particular, if $K > 0$ everywhere on Σ and $K_1 = 2\pi - K_0$ is chosen to be zero (i.e., the curves γ are those that bisect the total curvature of Σ); all extremals of (M, I, ϕ) must be regular. Supposing that γ is such an extremal, we have

$$\int_\gamma \kappa ds = \int_\gamma cK ds = 0,$$

and by the positivity of K, the constant c must be zero. As a result, $\kappa \equiv 0$ along γ. This proves the well-known theorem of Poincaré:

A smooth curve that bisects the total curvature of a compact convex surface must be a geodesic.

Further questions: It is pointed out in [Hsu92] that, even on a convex Riemannian surface Σ, the question of whether or not there exists a simple closed curve on Σ along which $\kappa = cK$ for some given constant c is subtle and worth further investigation. Also, noting that Hsu's theorem is only a sufficient condition for an extremal in (M, I, ϕ) to arise from Euler-Lagrange equations, one could ask whether or not the regularity condition rules out interesting extremals in M.

4. Bibliography

[Bryant87] R.Bryant, *On Notions of Equivalence for Variational Problems with one independent Variable*, Contemporary Math. **68** (1987) 65-76.

[Hsu92] L.Hsu, *Calculus of Variations via the Griffiths Formalism*, J. Diff. Geom. **36** (1992), 551-589.