CHAPTER 1

NORMED INVOLUTIVE ALGEBRAS

1.1. Involutive algebras

1.1.1. Let A be an algebra over the field $\mathbf{C}$ of complex numbers. An *involution* in A is a map $x \to x^*$ of A into itself such that

(i) $(x^*)^* = x$
(ii) $(x+y)^* = x^* + y^*$
(iii) $(\lambda x)^* = \bar{\lambda} x^*$
(iv) $(xy)^* = y^* x^*$

for any x, $y \in A$ and $\lambda \in \mathbf{C}$. An algebra over $\mathbf{C}$ endowed with an involution is called an *involutive algebra*. x^* is often called the *adjoint* of x. A subset of A which is closed under the involution operation is said to be *self-adjoint*. Property (i) implies that an involution in A is necessarily a bijection of A onto itself.

1.1.2. *Examples*

(1) On $A = \mathbf{C}$, the map $z \to \bar{z}$ (where $\bar{z}$ is the complex conjugate of z) is an involution with which A becomes a commutative involutive algebra.

(2) Let X be a locally compact space, and A the algebra of complex-valued continuous functions vanishing at infinity on X. Endowed with the map $f \to \bar{f}$, A is a commutative involutive algebra. When X is a single point, this reduces to example (1).

(3) Let H be a Hilbert space and $A = \mathscr{L}(H)$ the algebra of continuous endomorphisms of H. Furnished with the usual adjoint operation, A is an involutive algebra. Examples (2) and (3) will play a fundamental role. (Throughout the book, "Hilbert space" means "complex Hilbert space".)

(4) Let G be a unimodular locally compact group and A the convolution algebra $L^1(G)$. For each $f \in L^1(G)$ put $f^*(s) = \overline{f(s^{-1})}(s \in G)$. With the map $f \to f^*$, A is an involutive algebra.

1.1.3. We now introduce some terminology suggested by example (3) above. Let A be an involutive algebra. An element $x \in A$ is said to be *hermitian* if $x^* = x$ and *normal* if $xx^* = x^*x$. An idempotent hermitian element is called a projection. Each hermitian element is normal and the set of hermitian elements is a real vector subspace of A. If x and y are hermitian, we have $(xy)^* = y^*x^* = yx$, and so xy is hermitian if x and y commute. For every $x \in A$, xx^* and x^*x are hermitian, although a general hermitian element cannot be so represented as example (1) above shows.

1.1.4. Each $x \in A$ can be written uniquely in the form $x_1 + ix_2$ with x_1, x_2 hermitian. (In example (1) above, this expression is just the decomposition of a complex number into its real and imaginary parts.) In fact, if we put

$$x_1 = \frac{1}{2}(x + x^*), \qquad x_2 = \frac{1}{2\mathrm{i}}(x - x^*)$$

then x_1 and x_2 are hermitian and we have $x = x_1 + \mathrm{i}x_2$. Conversely, if $x = x_1 + \mathrm{i}x_2$ with x_1 and x_2 hermitian, we have $x^* = x_1 - \mathrm{i}x_2$ and so

$$x_1 = \frac{1}{2}(x + x^*), \qquad x_2 = \frac{1}{2\mathrm{i}}(x - x^*),$$

which proves our assertion. Note that

$$xx^* = x_1^2 + x_2^2 + \mathrm{i}(x_2x_1 - x_1x_2), \qquad x^*x = x_1^2 + x_2^2 - \mathrm{i}(x_2x_1 - x_1x_2)$$

so that x is normal if and only if x_1 and x_2 commute.

1.1.5. If A possesses a left identity 1, we have, for each $x \in A$,

$$x \cdot 1^* = (1 \cdot x^*)^* = x^{**} = x,$$

so that 1^* is a right identity, and thus $1 = 1^*$ is the identity for A. If x is an invertible element of A, then

$$(x^{-1})^*x^* = (xx^{-1})^* = 1^* = 1, \qquad x^*(x^{-1})^* = (x^{-1}x)^* = 1^* = 1,$$

so that x^* is invertible, and $(x^*)^{-1} = (x^{-1})^*$; conversely, if x^* is invertible, $x^{**} = x$ is also invertible. Since $(x - \lambda \cdot 1)^* = x^* - \bar{\lambda} \cdot 1$ for each $\lambda \in \mathbf{C}$, it follows that

$$\mathrm{Sp}_A x^* = \overline{\mathrm{Sp}_A x}.$$

(In every unital algebra B, the spectrum of an element x, denoted by

$\mathrm{Sp}_B x$ or simply $\mathrm{Sp}\, x$, is the set of scalars λ such that $x - \lambda \cdot 1$ is not invertible.)

An element $x \in A$ is said to be *unitary* if $xx^* = x^*x = 1$, or in other words if x is invertible and $x = x^{*-1}$. [In example (1) above, the unitary elements are the complex numbers of absolute value 1.] The unitary elements of A constitute a group under multiplication, the *unitary group* of A; in fact, if x and y are unitary elements of A, then

$$(xy)^{*-1} = (y^*x^*)^{-1} = x^{*-1}y^{*-1} = xy,$$

so that xy is unitary, and $(x^{-1})^{*-1} = (x^{*-1})^{-1} = x^{-1}$, so that x^{-1} is unitary.

1.1.6. Let A be an involutive algebra. If $\tilde{A}$ is the algebra obtained from A by adjoining an identity to A, it is easy to see that the involution on A can be extended to $\tilde{A}$ in a unique way: we define $(\lambda, x)^* = (\bar{\lambda}, x^*)$ for $\lambda \in \mathbf{C}$, $x \in A$. For each $x \in A$, we have

$$\mathrm{Sp}'_A x^* = \overline{\mathrm{Sp}'_A x}$$

(For any algebra B, unital or not, $\mathrm{Sp}'_B x$ or $\mathrm{Sp}' x$ denotes the set $\mathrm{Sp}_{\tilde{B}} x$, where $\tilde{B}$ is the algebra obtained from B by the adjunction of an identity. Clearly, $0 \in \mathrm{Sp}'_B x$ for each $x \in B$.)

1.1.7. Let A and B be two involutive algebras. A *morphism* (resp. *isomorphism*) of A into B is a map (resp. a bijection) φ of A into B such that $\varphi(x+y) = \varphi(x) + \varphi(y)$, $\varphi(\lambda x) = \lambda\varphi(x)$, $\varphi(xy) = \varphi(x)\varphi(y)$, $\varphi(x^*) = \varphi(x)^*$ for any $x, y \in A$, $\lambda \in \mathbf{C}$. In particular, φ is a morphism of the underlying algebra of A into the underlying algebra of B. In cases of possible confusion, we say more precisely "morphism for the involutive algebra structure" or "morphism for the algebra structure" as the case may be.

1.1.8. Let A be an involutive algebra. An involutive subalgebra of A is a subalgebra of A which is closed under the involution. The intersection of any family of involutive subalgebras is again an involutive subalgebra; thus, if M is any subset of A, there is a smallest involutive subalgebra of A containing M, namely the intersection of all the involutive subalgebras of A which contain M, and this is called the involutive subalgebra of A generated by M; it is the set of linear combinations of elements of the form $x_1x_2 \dots x_n$ where $x_1, x_2, \dots, x_n \in M \cup M^*$. When M consists of a single element x, this subalgebra is commutative if and only if x is normal.

Let A be an involutive algebra and B a self-adjoint two-sided ideal of A. The involution on A induces an involution on the quotient algebra A/B, and the canonical map of A onto A/B is a morphism.

The product of any family of involutive algebras is itself an involutive algebra in a natural way.

The reversed algebra of an involutive algebra is itself an involutive algebra when endowed with the same involution.

1.1.9. Let A be an involutive algebra and M a self-adjoint subset of A. Then the commutant M' of M in A is an involutive subalgebra of A, and the bicommutant M'' of M in A is thus an involutive subalgebra of A containing M; it is in general different from the involutive subalgebra of A generated by M (for example when A is commutative). If the elements of M commute pairwise, then $M \subset M'$, so that $M' \supset M''$ and M'' is commutative. If $x \in A$ and $M = \{x, x^*\}$, then M'' is commutative if and only if x is normal.

1.1.10. Let A be an involutive algebra. If f is a linear form on A, the function $x \to \overline{f(x^*)}$ on A is also a linear form on A, denoted by f^* and called the adjoint of f. Plainly, $f^{**} = f$, $(f + f')^* = f^* + f'^*$, $(\lambda f)^* = \bar{\lambda} f^*$ if $\lambda \in \mathbf{C}$, and f is said to be hermitian if $f = f^*$. Every linear form f on A has a unique expression of the form $f_1 + if_2$ with f_1, f_2 hermitian; indeed, we have

$$f_1 = \frac{1}{2}(f + f^*), \qquad f_2 = \frac{1}{2i}(f - f^*)$$

A linear form f on A is hermitian, if and only if f takes real values on the set A_h of hermitian elements of A. The map $f \to f \mid A_h$ is an isomorphism of the real vector space of hermitian forms onto the dual space of the real vector space A_h. If A is commutative and χ is a character of A, then χ^* is also a character of A.

References: [1101], [1323].

1.2. Normed involutive algebras

1.2.1. Definition. A *normed involutive algebra* is a normed algebra A together with an involution $x \to x^*$ such that $\|x^*\| = \|x\|$ for each $x \in A$. If, in addition, A is complete, A is called an *involutive Banach algebra*.

1.2.2. *Examples.* The four examples of 1.1.2 are examples of involutive Banach algebras if the norms are defined as follows: in example (1) set $\|z\| = |z|$ for each $z \in \mathbf{C}$; in example (2) set $\|f\| = \sup_{t\in X} |f(t)|$ for each $f \in A$; in example (3) take the usual norm of $\mathscr{L}(H)$; and in example (4), set $\|f\| = \int_G |f(g)|dg$ for each $f \in L^1(G)$.

1.2.3. Let A be a normed involutive algebra, and $\tilde{A}$ the involutive algebra obtained from A by the adjunction of an identity. The norm on A can be extended to $\tilde{A}$ in such a way as to make $\tilde{A}$ a normed involutive algebra (for example, one can put $\|(\lambda, x)\| = |\lambda| + \|x\|$ for $\lambda \in \mathbf{C}$, $x \in A$). Any normal involutive algebra so obtained is called a normed involutive algebra obtained from A by the adjunction of an identity.

1.2.4. Let A and B be two normed involutive algebras. A morphism of A into B will simply mean a morphism of the underlying involutive algebras, without any condition on the norms. On the other hand, an isomorphism will mean a *norm-preserving* isomorphism of the underlying involutive algebras.

1.2.5. The closure of an involutive subalgebra of a normed involutive algebra A is itself an involutive subalgebra. If $M \subset A$, the smallest closed involutive subalgebra B containing M is called the closed involutive subalgebra generated by M and is the closure of the involutive subalgebra generated by M. If M consists of a single normal element, then B is commutative.

The quotient of a normed involutive algebra by a closed self-adjoint two-sided ideal, the product of a finite number of normed involutive algebras, the reversed algebra of a normed involutive algebra and the completion of a normed involutive algebra are all normed involutive algebras in a natural way.

1.2.6. Let A be a normed involutive algebra. If f is a continuous linear form on A, then f^* is also continuous and $\|f^*\| = \|f\|$ as the unit ball of A is self-adjoint. The set A_h of hermitian elements of A is a real normed vector space. Now let f be a continuous hermitian linear form on A and let $g = f \mid A_h$. Then $\|f\| = \|g\|$; in fact, it is clear that $\|f\| \geq \|g\|$; on the other hand, for each $\epsilon > 0$ there exists an $x \in A$ such that $\|x\| \leq 1$ and $|f(x)| \geq \|f\| - \epsilon$ and multiplying x by a scalar of absolute value 1 if necessary, we can assume that $f(x) \geq 0$. Then

$$|g(\tfrac{1}{2}(x + x^*))| = \tfrac{1}{2}|f(x) + f(x^*)| = f(x) \geq \|f\| - \epsilon$$

and since $\|\frac{1}{2}(x+x^*)\| \leq 1$, we see that $\|g\| \geq \|f\| - \epsilon$ and the assertion follows. The continuous hermitian linear forms on A may thus be identified with the continuous real linear forms on A_h.

References: [1101], [1323].

1.3. C^*-algebras

1.3.1. Definition. A *C^*-algebra* is an involutive Banach algebra A such that $\|x\|^2 = \|x^*x\|$ for every $x \in A$.

1.3.2. Examples (1), (2) and (3) of 1.1.2, 1.2.2 are examples of C^*-algebras. Example (4), however, is not in general an example of a C^*-algebra.

1.3.3. If A is a C^*-algebra, so is every closed involutive subalgebra of A. In particular, if H is a Hilbert space, every closed involutive subalgebra of $\mathscr{L}(H)$ is a C^*-algebra; it will later be seen (2.6.1) that every C^*-algebra is isomorphic to a C^*-algebra of this type, and it is this example that has given rise to the theory of C^*-algebras.

Let $(A_i)_{i\in I}$ be a family of C^*-algebras. Let A be the set of $(x_i)_{i\in I}$ such that $x_i \in A_i$ for each $i \in I$ and such that $\sup_{i\in I} \|x_i\| < +\infty$. With the algebraic operations

$$(x_i)+(y_i)=(x_i+y_i), \qquad \lambda(x_i)=(\lambda x_i),$$
$$(x_i)(y_i)=(x_iy_i), \qquad (x_i)^*=(x_i^*),$$

and the norm

$$\|(x_i)\| = \sup \|x_i\|$$

we immediately see that A is a C^*-algebra, called the product C^*-algebra of the A_i's. It should be noted that the set A is not the product (set) of the A_i's.

Let A be a C^*-algebra, and consider the algebra obtained from A by replacing the multiplication $(x, y) \to xy$ by the multiplication $(x, y) \to yx$, while all other algebraic operations and the norm are the same as those of A. Consider, in other words, the reversed normed involutive algebra A^0 of A. Then it is immediate that A^0 is a C^*-algebra.

The question of quotient C^*-algebras is a more delicate matter (cf. 1.8.2).

1.3.4. Let A be a Banach algebra endowed with an involution such that

$$\|x\|^2 \leq \|x^*x\|.$$

It follows that $\|x\|^2 \leq \|x^*\| \cdot \|x\|$, hence that $\|x\| \leq \|x^*\|$ and interchanging x and x^*, we see that $\|x^*\| = \|x\|$. The above hypothesis thus implies that

$$\|x\|^2 \leq \|x^*x\| \leq \|x\|^2,$$

so that A is a C^*-algebra.

1.3.5. Let A be a C^*-algebra. For each $x \in A$, we have

$$\|x\| = \sup_{\|x'\| \leq 1} \|xx'\|.$$

In fact, it is clear that $\|x'\| \leq 1$ implies $\|xx'\| \leq \|x\|$. To show that $\|x\| \leq \sup_{\|x'\| \leq 1} \|xx'\|$, we can assume that $\|x\| = 1$; then $\|x^*\| = 1$ and

$$\sup_{\|x'\| \leq 1} \|xx'\| \geq \|xx^*\| = \|x\|^2 = 1.$$

1.3.6. Let A be a unital C^*-algebra. Then

$$\|1\|^2 = \|1^*1\| = \|1\|, \quad \text{so that} \quad \|1\| = 0 \text{ or } 1.$$

We thus see that, unless $A = 0$, $\|1\| = 1$, and it follows that if $A \neq 0$ then $\|u\| = \|u^*u\|^{1/2} = 1$ for each unitary element u of A.

1.3.7. We recall that no continuity condition appeared in the definition of morphisms of normed involutive algebras. We shall see that, for C^*-algebras, a morphism is automatically continuous, or, more precisely, that we have:

PROPOSITION. *Let A be an involutive Banach algebra, B a C^*-algebra and π a morphism of A into B. Then $\|\pi(x)\| \leq \|x\|$ for every $x \in A$.*

For each hermitian element y of B, we have $\|y^2\| = \|y^*y\| = \|y\|^2$ and hence by induction $\|y^{2^n}\|^{2^{-n}} = \|y\|$. As $n \to +\infty$ the left hand side of this equation tends to the spectral radius $\rho(y)$ of y (cf. B1), so that

(1) $$\rho(y) = \|y\|.$$

Now for each $x \in A$, we have

$$\mathrm{Sp}'_B \pi(x) \subseteq \mathrm{Sp}'_A x \quad \text{so that} \quad \rho(\pi(x)) \leq \rho(x) \leq \|x\|,$$

and hence, using (1), we have

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = \rho(\pi(x^*x)) \leq \|x^*x\| \leq \|x^*\| \cdot \|x\| = \|x\|^2.$$

1.3.8. The following proposition will enable us almost always to confine our attention to unital C^*-algebras without any loss of generality.

PROPOSITION. *Let A be a C^*-algebra and $\tilde{A}$ the involutive algebra obtained from A by the adjunction of an identity. Then the norm on A can be extended to $\tilde{A}$ in exactly one way that makes $\tilde{A}$ a C^*-algebra.*

The uniqueness follows from 1.3.7. We prove the existence. First suppose that A has an identity e and let 1 denote the identity of $\tilde{A}$. In $\tilde{A}$, A and $\mathbf{C}(1-e)$ are complementary self-adjoint two-sided ideals, and so there is an isomorphism of the involutive algebra $\tilde{A}$ onto the involutive algebra $\mathbf{C} \times A$ which maps A onto $\{0\} \times A$; $\mathbf{C} \times A$ can then be given the product C^*-algebra structure. Now suppose that A does not possess an identity. Each element x of $\tilde{A}$ defines an operator of left multiplication L_x in the two-sided ideal A of $\tilde{A}$; put $\|x\| = \|L_x\|$. If $x \in A$, this coincides with the original norm on A, by 1.3.5. Moreover, it is clear that $x \to \|x\|$ is a seminorm on $\tilde{A}$ and that $\|xy\| \leq \|x\| \cdot \|y\|$. This seminorm is in fact a norm, for let $x = \lambda - x'$ ($\lambda \in \mathbf{C}$, $x' \in A$) be an element of $\tilde{A}$ such that $\|L_x\| = 0$, i.e. such that $xy = 0$ for each $y \in A$. We show that then $x = 0$. If $\lambda \neq 0$, we have for each $y \in A$, $0 = \lambda^{-1}xy = y - \lambda^{-1}x'y$, $\lambda^{-1}x'$ is a left identity for A, and A possesses an identity (1.1.5) contrary to hypothesis. Hence $\lambda = 0$, so that $x \in A$, and $\|x\| = 0$ implies that $x = 0$; it follows that $x \to \|x\|$ is a norm on $\tilde{A}$. Since A is complete and of codimension 1 in $\tilde{A}$, $\tilde{A}$ is also complete. It remains to show (1.3.4) that $\|x\|^2 \leq \|x^*x\|$ for each $x \in \tilde{A}$ and it is enough to do this for $\|x\| = 1$. For each $r < 1$, there is $y \in A$ such that $\|y\| \leq 1$ and $\|xy\|^2 \geq r$; then, as $xy \in A$, we have $\|x^*x\| \geq \|y^*(x^*x)y\| = \|(xy)^*xy\| = \|xy\|^2 \geq r$ and therefore $\|x^*x\| \geq 1$.

1.3.9. PROPOSITION. *Let A be a C^*-algebra.*

(i) *If h is a hermitian element of A, each element of $\mathrm{Sp}'h$ is real.*

(ii) *If A has an identity, and u is a unitary element of A, each element of $\mathrm{Sp}\, u$ has absolute value 1.*

In proving both parts of this result we may assume, by 1.3.8, that A has an identiy (and that $A \neq 0$). We have $\|u\| = \|u^{-1}\| = 1$ by 1.3.6, so that $\rho(u) \leq 1$, $\rho(u^{-1}) \leq 1$ and $\mathrm{Sp}\, u$ and $\mathrm{Sp}(u^{-1}) = (\mathrm{Sp}\, u)^{-1}$ are both contained in the unit disc of the complex plane $\mathbf{C}$, from which (ii) follows immediately. For (i)

$$(\exp(\mathrm{i}h))^* = \left(1 + \mathrm{i}h + \frac{\mathrm{i}^2h^2}{2!} + \cdots\right)^* = 1 + (-\mathrm{i})h + \frac{(-\mathrm{i})^2h^2}{2!} + \cdots$$
$$= \exp(-\mathrm{i}h),$$

so that $\exp(\mathrm{i}h)$ is unitary; thus if $z \in \operatorname{Sp} h$, we have $|\exp(\mathrm{i}z)| = 1$ (B 4) and so $z \in \mathbf{R}$. (Throughout the book, $\mathbf{R}$ denotes the set of real numbers.)

1.3.10. The following proposition shows that, for C^*-algebras, it does not matter which algebra containing a particular element we take in the definition of the spectrum of that element.

PROPOSITION. *Let A be a C^*-algebra, B a sub-C^*-algebra and x an element of B. Then*

(i) $\operatorname{Sp}'_A x = \operatorname{Sp}'_B x$.

(ii) *If A has an identity which also lies in B, then* $\operatorname{Sp}_A x = \operatorname{Sp}_B x$.

We see that (i) follows from (ii) on adjoining an identity. We therefore prove (ii). If x is hermitian, we have $\operatorname{Sp}_B x \subseteq \mathbf{R}$ (1.3.9), so that $\operatorname{Sp}_B x = \operatorname{Sp}_A x$ (B 2). In the general case, if $x \in B$ is invertible in A, xx^* is also invertible in A, therefore in B by the above, and so x has a right inverse in B; similarly x has a left inverse in B and is therefore invertible in B. Applying this result to $x - \lambda \cdot 1$ where $\lambda \in \mathbf{C}$, we obtain (ii).

C^*-algebras are called completely regular algebras in [1101] and B^*-algebras by numerous authors.

References: [604], [1101], [1320], [1322], [1323].

1.4. Commutative C^*-algebras

1.4.1. The theory of the above is summed up in the following theorem:

THEOREM. *Let A be a commutative C^*-algebra, S its spectrum (which is a locally compact space), and B the C^*-algebra of continuous complex-valued functions on S which vanish at infinity. Then*

(i) *Every character of A is hermitian.*

(ii) *The Gelfand map is an isomorphism of the C^*-algebra A onto the C^*-algebra B.*

Let χ be a character of A. If y is an hermitian element of A, we have $\chi(y) \in \operatorname{Sp}' y \subset \mathbf{R}$ (1.3.9). For an arbitrary element x of A write $x = x_1 + \mathrm{i}x_2$

with x_1, x_2 hermitian; we have

$$\chi(x^*) = \chi(x_1 - ix_2) = \chi(x_1) - i\chi(x_2) = \overline{\chi(x)}$$

which proves (i).

In other words, if $\mathcal{F}$ denotes the Gelfand map, then $\mathcal{F}(x^*) = \overline{\mathcal{F}(x)}$ for each $x \in A$. Moreover, $\mathcal{F}(A)$ separates the points of S and for each point of S there is at least one function belonging to $\mathcal{F}(A)$ which does not vanish there (B 3). The Stone-Weierstrass theorem then shows that $\mathcal{F}(A)$ is dense in B. The proof will be complete if we now show that $\mathcal{F}$ is isometric. Now, $\|\mathcal{F}(y)\| = \|y\|$ for any hermitian y [1.3.7, formula (1)], and so for each $x \in A$,

$$\|x\|^2 = \|x^*x\| = \|\mathcal{F}(x^*x)\| = \|\overline{\mathcal{F}(x)} \cdot \mathcal{F}(x)\| = \|\mathcal{F}x\|^2.$$

1.4.2. Retaining the above notation, if $f \in B$, the symbol $g(f)$ is well defined for each continuous complex-valued function g on $f(S) \cup \{0\} = \mathrm{Sp}'f$ such that $g(0) = 0$, and represents an element of B. Thus, if $x \in A$, it is possible to define $h(x) \in A$ for any continuous complex valued h defined on $\mathrm{Sp}'x$ and satisfying $h(0) = 0$. We shall see in 1.5 that one can define such a "functional calculus" for the normal elements of any, not necessarily commutative, C^*-algebra, and this turns out to be a very useful tool in the sequel.

1.4.3. PROPOSITION. *Let A be a commutative unital C^*-algebra, S its spectrum and let $x \in A$. Suppose that the sub-C^*-algebra of A generated by 1 and x is equal to A. Then $\chi \to \chi(x)$ is a homeomorphism of S onto $\mathrm{Sp}_A x$.*

This map is continuous and its range is $\mathrm{Sp}_A x$ (B 3). Moreover, let χ, $\chi' \in S$ be such that $\chi(x) = \chi'(x)$. Since χ and χ' are continuous and hermitian, the set A' of those $y \in A$ for which $\chi(y) = \chi'(y)$ is a sub-C^*-algebra of A containing x. Hence $A' = A$ and $\chi' = \chi$. The map under consideration is injective, and is therefore a homeomorphism as S is compact.

References: [100], [618], [1101], [1323].

1.5. Functional calculus in C^*-algebras

1.5.1. THEOREM. *Let A be a unital C^*-algebra, x a normal element of A, $S = \mathrm{Sp}_A x$ and A' the C^*-algebra of continuous complex-valued functions on S. Then there is a unique morphism ϕ of A' into A such that*

$\phi(1) = 1$, and $\phi(\iota) = x$ where ι denotes the function $z \to z$ on S. Moreover, this morphism is isometric and its image $\phi(A')$ is the sub-C^*-algebra of A generated by 1 and x and therefore consists entirely of normal elements.

The polynomials in z and $\bar{z}$ are dense in A' and each morphism of A' into A is continuous (1.3.7) which establishes the uniqueness of ϕ. Let B be the commutative sub-C^*-algebra of A generated by 1 and x, T its spectrum, C the C^*-algebra of continuous complex-valued functions on T and $\mathcal{F}_B$ the Gelfand map for B, which is an isomorphism of B onto C (1.4.1). Proposition 1.4.3 furnishes a homeomorphism of T onto $\mathrm{Sp}_B x = S$ (1.3.10) which induces an isomorphism ψ: $A' \to C$ that maps ι to $\mathcal{F}_B x$, since we have, for each $\chi \in T$, $(\mathcal{F}_B x)(\chi) = \chi(x) = \iota(\chi(x))$. Now consider the composite isomorphism

$$A' \xrightarrow{\psi} C \xrightarrow{\mathcal{F}_B^{-1}} B$$

Composing this with the canonical injection of B into A, we obtain a morphism of A' into A with the properties required by the theorem.

1.5.2. Definition. Let A be a unital C^*-algebra. If x is a normal element of A and if f is a continuous complex valued function on $\mathrm{Sp}_A x$, the element $\phi(f)$ of theorem 1.5.1 is denoted by $f(x)$.

The fact that ϕ is an isometric isomorphism is expressed in the following formulae in which f and g denote continuous complex-valued functions on $\mathrm{Sp}_A x$:

(1) $$(f+g)(x) = f(x) + g(x),$$

(2) $$(fg)(x) = f(x)g(x),$$

(3) $$\bar{f}(x) = f(x)^*,$$

(4) $$\|f(x)\| = \|f\|.$$

If f is the restriction to $\mathrm{Sp}_A x$ of a polynomial $z \to P(z, \bar{z})$ in z and $\bar{z}$, then $f(x) = P(x, x^*)$ where $P(x, x^*)$ has its usual algebraic interpretation (remember that $xx^* = x^*x$).

With the above notation, we have

$$\mathrm{Sp}_A f(x) = \mathrm{Sp}_B f(x) = \mathrm{Sp}_{A'} f = f(S),$$

or in other words,

(5) $$\mathrm{Sp}_A f(x) = f(\mathrm{Sp}_A x).$$

1.5.3. PROPOSITION. *Let A and B be unital C^*-algebras, ϕ a morphism of A into B mapping 1 to 1 and x a normal element of A so that $\phi(x)$ is a normal element of B. Let f be a continuous complex-valued function on $\mathrm{Sp}_A x$. Then if the restriction of f to $\mathrm{Sp}_B \phi(x)$ is again denoted by f, we have $\phi(f(x)) = f(\phi(x))$.*

Let C be the C^*-algebra of continuous complex-valued functions on $\mathrm{Sp}_A x$. The maps $f \to \phi(f(x))$ and $f \to f(\phi(x))$ are morphisms of C into B which take the same value when f is any one of the functions $z \to 1$, $z \to z$, $z \to \bar{z}$. Since the sub-C^*-algebra of C generated by these functions is equal to the whole of C, the two morphisms are identical.

1.5.4. COROLLARY. *Let A be a commutative unital C^*-algebra, x an element of A, $\mathcal{G}$ the Gelfand map for A, and f a continuous complex-valued function on $\mathrm{Sp}_A x$. Then $\mathcal{G}(f(x)) = f \circ \mathcal{G}(x)$.*

This follows, for instance, from 1.5.3 applied to $\phi = \mathcal{G}$.

1.5.5. COROLLARY. *Let A be a unital C^*-algebra, x a normal element of A, C the C^*-algebra of continuous complex-valued functions on $\mathrm{Sp}\, x$, f an element of C, C' the C^*-algebra of continuous complex-valued functions on $\mathrm{Sp}\, f(x) = f(\mathrm{Sp}\, x)$, and g an element of C'. Then $g \circ f \in C$ and $(g \circ f)(x) = g(f(x))$.*

The map $g \to (g \circ f)(x)$ is a morphism of C' into A which maps 1 to 1 and the function $z \to z$ to $f(x)$. From the uniqueness statement of theorem 1.5.1, it follows that $(g \circ f)(x) = g(f(x))$.

1.5.6. PROPOSITION. *Let A be a C^*-algebra, x a normal element of A, $S = \mathrm{Sp}' x$ and A' the C^*-algebra of continuous complex-valued functions on S which vanish at 0. Then there is exactly one morphism ϕ of A' into A such that $\phi(\iota) = x$ where ι is the function $z \to z$ on S. This morphism is isometric and its image $\phi(A')$ is the sub-C^*-algebra of A generated by x which therefore consists entirely of normal elements.*

Since the polynomials in z and $\bar{z}$ without constant term are dense in A', the uniqueness of ϕ is immediate. The existence follows from theorem 1.5.1 on adjoining an identity to A.

1.5.7. By the adjunction of an identity, all the results of this section can be extended immediately, with obvious modifications, to the case of non-unital C^*-algebras. We shall thus make use of these results even when an identity is not assumed to be present.

In particular, let x be an hermitian element of the C^*-algebra A, so that its spectrum is real. Consider the continuous functions of a real variable

$$t \to f_1(t) = \sup(t, 0), \qquad t \to f_2(t) = \sup(-t, 0), \qquad t \to f_3(t) = |t|.$$

We put $x^+ = f_1(x)$, $x^- = f_2(x)$, $|x| = f_3(x)$. These are hermitian elements of A, (and, indeed, of the sub-C^*-algebra of A generated by x), because $\bar{f}_1 = f_1$, $\bar{f}_2 = f_2$, $\bar{f}_3 = f_3$. Since f_1, f_2, f_3 take non-negative values only, we have

(1) $$\mathrm{Sp}'(x^+) \geq 0, \qquad \mathrm{Sp}'(x^-) \geq 0, \qquad \mathrm{Sp}'(|x|) \geq 0.$$

Since

$$f_1(t) - f_2(t) = t, \qquad f_1(t) + f_2(t) = |t| \quad \text{and} \quad f_1(t) f_2(t) = 0$$

we have

(2) $$x = x^+ - x^-, \qquad |x| = x^+ + x^-, \qquad x^+ x^- = x^- x^+ = 0.$$

The norm of $|x|$ is the same as that of x, while the norms of x^+ and x^- are less than or equal to that of x. We note, with a view to no. 1.6.4, that x^+ and x^- are the squares of two hermitian elements whose product is zero: consider the functions $\sqrt{f_1}$ and $\sqrt{f_2}$.

1.5.8. Let A be a C^*-algebra. For each positive integer n, we have $A = A^n$, the set of linear combinations of products of n elements of A; it is enough to show that each hermitian element x of A is a product of n elements of A. Now, if $f_1, \ldots, f_n$ are continuous real-valued functions of a real variable such that

$$f_1(t) f_2(t) \cdots f_n(t) = t, \qquad f_1(0) = \cdots = f_n(0) = 0,$$

then

$$x = f_1(x) f_2(x) \cdots f_n(x).$$

References: [618], [1101], [1323].

1.6. Positive elements in C^*-algebras

1.6.1. PROPOSITION. *Let A be a C^*-algebra and x an hermitian element of A. Then the following conditions are equivalent:*

(i) $\mathrm{Sp}'_A x \geq 0$.

(ii) *x is of the form yy^* for some $y \in A$.*

(iii) *x is of the form h^2 for some hermitian $h \in A$.*

Furthermore, the set P of those elements which satisfy these conditions is a closed convex cone such that $P \cap (-P) = \{0\}$.

To prove this, we first denote by P the set of those hermitian elements of A which satisfy condition (i).

(i)$\Rightarrow$(iii): if $\mathrm{Sp}'_A x \geq 0$, $h = x^{1/2}$ exists as a hermitian element of A and we have $x = h^2$.

(iii)$\Rightarrow$(i): if $x = h^2$ with h hermitian, we have $\mathrm{Sp}'_A x = (\mathrm{Sp}'_A h)^2 \geq 0$ since $\mathrm{Sp}'_A h$ is real.

(iii)$\Rightarrow$(ii): obvious.

To prove the implication (ii)$\Rightarrow$(iii) and the last part of the proposition we need the following lemmas:

1.6.2. LEMMA. *Suppose that A is unital. If $x \in A$ is hermitian and $\|1 - x\| \leq 1$ then $x \in P$. If $x \in P$ and $\|x\| \leq 1$ then $\|1 - x\| \leq 1$.*

By passing to the sub-C^*-algebra of A generated by x, it is enough to consider the case of commutative A which can in turn, thanks to 1.4.1, be reduced to the case where A is the C^*-algebra of continuous complex-valued functions on a compact space. The lemma is then clear.

1.6.3. LEMMA. *Suppose that A is unital and let x be an hermitian element of A. In order that $x \in P$, it is necessary and sufficient that $\|(\|x\| \cdot 1) - x\| \leq \|x\|$.*

We can assume that $x \neq 0$, and then, multiplying by an appropriate scalar, that $\|x\| = 1$. Lemma 1.6.3 then follows at once from Lemma 1.6.2.

1.6.4. We now return to the situation of 1.6.1. In showing that P is a closed convex cone such that $P \cap (-P) = \{0\}$ we can assume, thanks to 1.3.8, that A is unital. Lemma 1.6.3 implies that P is closed. It is clear that $x \in P$ and $\lambda \geq 0$ imply that $\lambda x \in P$. We now show that $x + y \in P$ whenever $x, y \in P$. We can assume that $\|x\| \leq 1$ and $\|y\| \leq 1$. Then $\|1 - x\| \leq 1$ and $\|1 - y\| \leq 1$ (1.6.2), and so $\|1 - \frac{1}{2}(x + y)\| = \frac{1}{2}\|1 - x + 1 - y\| \leq \frac{1}{2}(\|1 - x\| + \|1 - y\|) \leq 1$ so that $\frac{1}{2}(x + y) \in P$ (1.6.2) and thus $x + y \in P$. If $x \in P \cap (-P)$ then $\mathrm{Sp}\, x = \{0\}$, $\rho(x) = 0$ and so $x = 0$ [1.3.7, formula (1)].

We finally prove the implication (ii)$\Rightarrow$(iii) of 1.6.1 without assuming that A is unital. Let $y \in A$, and write $(yy^*)^+ = u^2$, $(yy^*)^- = v^2$ with u, v hermitian elements of A such that $uv = 0$ (1.5.7). Then

$$(vy)(vy)^* = v(yy^*)v = vu^2v - v^4 = -v^4 \in -P,$$

since (iii)$\Rightarrow$(i). Let $vy = s + it$ with s, t hermitian. We have

$$\begin{aligned}(vy)^*(vy) &= -(vy)(vy)^* + (s+it)(s-it) + (s-it)(s+it)\\ &= -(vy)(vy)^* + 2s^2 + 2t^2 \in P\end{aligned}$$

as $-(vy)(vy)^* \in P$, $s^2 \in P$, $t^2 \in P$ and P is a convex cone. Hence $(vy)(vy)^* \in P$ because, in any algebra, the spectrum of a product is independent of the order of its factors (B 26). Hence $(vy)(vy)^* \in P \cap (-P) = 0$. Thus $v^4 = 0$ and so $v = 0$ and $yy^* = u^2$. Proposition 1.6.1 is now proved.

1.6.5. Definition. Let A be a C^*-algebra. An element $x \in A$ is said to be *positive* and we write $x \geq 0$ if it is hermitian and satisfies the three equivalent conditions of proposition 1.6.1. The set of positive elements of A is denoted by A^+.

If B is a sub-C^*-algebra of A and $x \in B$, it follows from 1.3.10 that the statement $x \geq 0$ has the same meaning whether it is interpreted in A or in B.

Since A^+ is a convex cone and $A^+ \cap (-A^+) = \{0\}$, the relation $x - y \geq 0$ is a partial ordering in A which is compatible with the real vector space structure of A; we write this relation $x \geq y$ or $y \leq x$. If z is an hermitian element of A then $z^+ \geq 0$, $z^- \geq 0$ and $|z| \geq 0$ by 1.5.7, formulae (1); z^+ (resp. z^-) is called the positive (resp. negative) part of z. By 1.5.7, formulae (2), each hermitian element of A is the difference of two elements of A^+.

1.6.6. Let A be the C^*-algebra of continuous complex-valued functions vanishing at infinity on a locally compact space T and suppose $f \in A$. It follows from condition (1) of proposition 1.6.1 and the fact that $\mathrm{Sp}'_A f = f(T) \cup \{0\}$ that the relation $f \geq 0$ has its usual meaning in the algebra A.

1.6.7. Let H be a Hilbert space, A the C^*-algebra $\mathscr{L}(H)$, and x an element of A. We show that the condition $x \geq 0$ is equivalent to the condition $(x\xi \mid \xi) \geq 0$ for every $\xi \in H$, i.e. to the usual definition of positive operators. If $x = y^*y$ for some $y \in A$, we have

$$(x\xi \mid \xi) = (y^*y\xi \mid \xi) = \|y\xi\|^2 \geq 0 \quad \text{for each } \xi \in H.$$

Conversely, suppose that $(x\xi \mid \xi) \geq 0$ for each $\xi \in H$. For every $\eta \in H$, we have

$$\begin{aligned} 0 &\leq (x(x^-\eta) \mid x^-\eta) \\ &= ((x^+ - x^-)(x^-\eta) \mid x^-\eta) \\ &= -(x^-x^-\eta \mid x^-\eta) = -((x^-)^3\eta \mid \eta). \end{aligned}$$

Since $(x^-)^3 \geq 0$ we also have $((x^-)^3\eta \mid \eta) \geq 0$ and thus $((x^-)^3\eta \mid \eta) = 0$, $(x^-)^3 = 0$, $x^- = 0$ and finally $x = x^+ \geq 0$.

1.6.8. Let A be a C^*-algebra, and a, b, x elements of A. If $a \leq b$, then $x^*ax \leq x^*bx$, as there is $y \in A$ such that $b - a = y^*y$ from which it follows that

$$x^*bx - x^*ax = x^*(y^*y)x = (yx)^*(yx) \geq 0.$$

Now suppose that A is unital and let b be an element of A with $b \geq 1$ (hence b is invertible). Applying the above with $x = b^{-1/2}$ we see that $1 \geq b^{-1}$. More generally, let a and b be invertible elements of A^+ such that $0 \leq a \leq b$. Then $a^{-1} \geq b^{-1}$; indeed, from the above, we have that $1 \leq a^{-1/2}ba^{-1/2}$, $1 \geq a^{1/2}b^{-1}a^{1/2}$ and so $a^{-1} \geq b^{-1}$.

1.6.9. Let A be a C^*-algebra, and let x, $y \in A^+$ be such that $y \leq x$. Then $\|y\| \leq \|x\|$. In fact, we can assume A to be unital and we then have $x \leq \|x\| \cdot 1$, for example by 1.4.1. Hence $0 \leq y \leq \|x\| \cdot 1$ and so $\|y\| \leq \|x\|$, again by 1.4.1.

1.6.10. Let A and B be two C^*-algebras and ϕ a morphism of A into B. It is plain that $\phi(A^+) \subset \phi(A) \cap B^+$. Suppose conversely that $y \in \phi(A) \cap B^+$; there exists $x \in A$ with $y = \phi(x)$ and we have $y = (y^*y)^{1/2} = \phi((x^*x)^{1/2})$, so that $y \in \phi(A^+)$. Hence $\phi(A^+) = \phi(A) \cap B^+$.

References: [604], [918], [1101], [1323], [1477].

1.7. Approximate identities in C^*-algebras

1.7.1. Let A be a C^*-algebra. We say that an approximate identity (u_λ) of A (B 29) is increasing if $u_\lambda \geq 0$ for every λ and if $\lambda \leq \mu$ implies $u_\lambda \leq u_\mu$.

1.7.2. PROPOSITION. *Let A be a C^*-algebra, and m a two-sided ideal of A which is dense in A. Then there is an increasing approximate identity of A consisting of elements of m. If A is separable, this approximate identity can be taken to be indexed by $\{1, 2, \ldots\}$.*

Let $\tilde{A}$ be the C^*-algebra obtained by adjoining an identity to A. Let Λ be the set of finite subsets of m ordered by inclusion. For $\lambda = \{x_1, \ldots, x_n\} \in \Lambda$, put

$$v_\lambda = x_1x_1^* + \cdots + x_nx_n^* \in m \quad \text{and} \quad u_\lambda = v_\lambda\left(\frac{1}{n} + v_\lambda\right)^{-1}$$

(the element u_λ is computed in $\tilde{A}$, but in fact $u_\lambda \in m$). Since the function of a real variable $t \to t(1/n + t)^{-1}$ only takes values between 0 and 1 for $t \geq 0$, we have $0 \leq u_\lambda \leq 1$. Furthermore,

$$\sum_{i=1}^{n} [(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* = (u_\lambda - 1)v_\lambda(u_\lambda - 1) = \frac{1}{n^2} v_\lambda\left(\frac{1}{n} + v_\lambda\right)^{-2}.$$

Now the function of a real variable $t \to t(1/n + t)^{-2}$ is always $\leq \frac{1}{4}n$. Thus

$$\sum_{i=1}^{n} [(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* \leq \frac{1}{4n}.$$

For $i = 1, 2, \ldots, n$ we deduce that

$$[(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* \leq \frac{1}{4n},$$

from which it follows that $\|(u_\lambda - 1)x_i\|^2 \leq \frac{1}{4}n$ (1.6.9). Thus $\|(u_\lambda - 1)x\| \to 0$ for each $x \in m$ and therefore for each $x \in A$ as $\bar{m} = A$ and $\|u_\lambda\| \leq 1$; hence

$$\|xu_\lambda - x\| = \|u_\lambda x^* - x^*\| \to 0.$$

(u_λ) is thus an approximate identity. Now let $\lambda, \mu \in \Lambda$ be such that $\lambda \leq \mu$. We have $\lambda = \{x_1, \ldots, x_n\}$, $\mu = \{x_1, \ldots, x_p\}$ with $p \geq n$, so that $v_\lambda \leq v_\mu$, and $(1/n + v_\lambda)^{-1} \geq (1/n + v_\mu)^{-1}$ by 1.6.8. For any real number $t \geq 0$, we have

$$\frac{1}{n}\left(\frac{1}{n} + t\right)^{-1} \geq \frac{1}{p}\left(\frac{1}{p} + t\right)^{-1},$$

so that

$$\frac{1}{n}\left(\frac{1}{n} + v_\mu\right)^{-1} \geq \frac{1}{p}\left(\frac{1}{p} + v_\mu\right)^{-1},$$

all of which implies that

$$1 - \frac{1}{n}\left(\frac{1}{n} + v_\lambda\right)^{-1} \leq 1 - \frac{1}{n}\left(\frac{1}{n} + v_\mu\right)^{-1} \leq 1 - \frac{1}{p}\left(\frac{1}{p} + v_\mu\right)^{-1},$$

i.e. $u_\lambda \leq u_\mu$. Hence the approximate identity (u_λ) is increasing.

Now suppose A is separable, so that there is a sequence $(y_1, y_2, \dots)$ which is dense in m. Put $u_n = u_{\{y_1,\dots,y_n\}}$. The above argument shows that, for each i, $\|u_n y_i - y_i\| \to 0$ as $n \to +\infty$. Since $\|u_n\| \leq 1$, we deduce that $u_n x \to x$ for each $x \in A$ and the proof is now concluded in the same way.

1.7.3. Let A be a C^*-algebra and I a right ideal of A. Then there exists a family (u_λ) in $I \cap A^+$ indexed by a directed set such that (1) $\|u_\lambda\| \leq 1$; (2) $\lambda \leq \mu$ implies $u_\lambda \leq u_\mu$; (3) for each $x \in \bar{I}$, $\|u_\lambda x - x\| \to 0$. The proof of 1.7.2 applies here unchanged.

References: [452], [1455].

1.8. Quotient of a C^*-algebra

1.8.1. Proposition. *Let A be a C^*-algebra, B a normed involutive algebra and ϕ an injective morphism of A into B. Then $\|\phi(x)\| \geq \|x\|$ for each $x \in A$.*

Let $x \in A$. If we can show that $\|\phi(x^*x)\| \geq \|x^*x\|$, we can deduce that

$$\|x\|^2 = \|x^*x\| \leq \|\phi(x^*x)\| = \|\phi(x^*)\phi(x)\| \leq \|\phi(x)\|^2,$$

from which the proposition follows. We can thus assume in addition that x is hermitian, and replacing ϕ by its restriction to the sub-C^*-algebra of A generated by x, we can assume that A is commutative. Replacing B by $\phi(A)$ we can assume B is commutative, and we then replace B by its completion. Furthermore, we can adjoin identities to A and B. In short we may confine attention to the case in which A and B are commutative, complete and unital. Now let S and T be the spectra of A and B, which are compact spaces. For each $\chi \in T$, $\chi \circ \phi$ is a character of A, i.e. an element of S, which we denote by $\phi'(\chi)$; if $x \in A$, then $\phi'(\chi)(x) = \chi(\phi(x))$ is a continuous function of χ; ϕ' is thus a continuous map of T into S and $\phi'(T)$ is a compact subset of S. If $\phi'(T) \neq S$, there exists a continuous complex-valued function f on S such that $f \neq 0$ and $f \mid \phi'(T) = 0$. By 1.4.1, f is the Gelfand transform of an $x \in A$. We have $x \neq 0$, $\chi(\phi(x)) = 0$ for every $\chi \in T$, hence $\phi(x) = 0$, and this is absurd.

1.8.2. Proposition. *Let A be a C^*-algebra and I a closed two-sided ideal of A. Then I is self-adjoint and A/I, endowed with the natural involutive algebra structure and the quotient norm, is a C^*-algebra.*

Let (u_λ) be a family with the properties of 1.7.3. If $x \in I$, we have

$$\|x^*u_\lambda - x^*\| = \|u_\lambda x - x\| \to 0,$$

and $x^*u_\lambda \in I$, so that $x^* \in \bar{I} = I$ and hence $I = I^*$.

We know that A/I is an involutive algebra and satisfies the axioms for a Banach algebra. Denote by $x \to \dot{x}$ the canonical map of A onto A/I. To show that A/I is a C^*-algebra, it is enough to show that $\|\dot{x}\|^2 \leqslant \|\dot{x}\dot{x}^*\|$ (1.3.4). We have

$$\|\dot{x}\| = \lim \|x - u_\lambda x\| \tag{1}$$

In fact, if $y \in I$ we have $u_\lambda y - y \to 0$, and so

$$\begin{aligned}\overline{\lim}\,\|x - u_\lambda x\| &= \overline{\lim}\,\|x - u_\lambda x + y - u_\lambda y\|\\ &= \overline{\lim}\,\|(1 - u_\lambda)(x + y)\| \leqslant \|x + y\|\end{aligned}$$

(we are working in $\tilde{A}$). Thus

$$\|\dot{x}\| \geqslant \overline{\lim}\,\|x - u_\lambda x\| \geqslant \underline{\lim}\,\|x - u_\lambda x\| \geqslant \inf_{y \in I} \|x + y\| = \|\dot{x}\|,$$

which proves (1). This established, we have for each $z \in I$,

$$\begin{aligned}\|\dot{x}\|^2 = \lim \|x - u_\lambda x\|^2 &= \lim \|(x - u_\lambda x)(x - u_\lambda x)^*\|\\ &= \lim \|xx^* - u_\lambda xx^* - xx^*u_\lambda + u_\lambda xx^*u_\lambda\|\\ &= \lim \|xx^* + z - u_\lambda z - u_\lambda xx^* - xx^*u_\lambda - zu_\lambda + u_\lambda zu_\lambda + u_\lambda xx^*u_\lambda\|\\ &= \lim \|(1 - u_\lambda)(xx^* + z)(1 - u_\lambda)\| \leqslant \|xx^* + z\|,\end{aligned}$$

and thus $\|\dot{x}\|^2 \leqslant \|\dot{x}\dot{x}^*\|$.

1.8.3. Corollary. *Let A and B be C^*-algebras, ϕ a morphism of A into B and I the kernel of ϕ. Consider the canonical decomposition of ϕ:*

$$A \to A/I \xrightarrow{\psi} \phi(A) \to B.$$

Then I is closed in A, $\phi(A)$ is closed in B and ψ is an (isometric) isomorphism of the C^*-algebra A/I onto the C^*-algebra $\phi(A)$.

Since ϕ is continuous (1.3.7), I is closed and A/I is a C^*-algebra (1.8.2). The morphism $A/I \to B$ obtained from ϕ by passing to the quotient is injective and therefore isometric (1.3.7 and 1.8.1). Hence $\phi(A)$ is complete and consequently closed in B.

1.8.4. COROLLARY. *Let A be a C^*-algebra, B a sub-C^*-algebra of A and I a closed two-sided ideal of A. Then $B+I$ is a sub-C^*-algebra of A and the C^*-algebras $(B+I)/I$ and $B/B\cap I$ are canonically isomorphic.*

Let ϕ be the canonical morphism $A\to A/I$, and let ψ be the restriction of ϕ to B. Then $\psi(B)=(B+I)/I$ is closed in A/I (1.8.3), and so $B+I$ is closed in A. Consider the canonical decomposition of ψ:

$$B\to B/(B\cap I)\to\psi(B)\to A/I.$$

By (1.8.3), the morphism $B/(B\cap I)\to\psi(B)=(B+I)/I$ is a C^*-algebra isomorphism.

1.8.5. PROPOSITION. *Let A be a C^*-algebra, I a closed two-sided ideal of A and J a closed two-sided ideal of I. Then J is a closed two-sided ideal of A.*

We have $J=J^3$ (1.5.8), and so $AJA=AJ^3A\subset IJI\subset J$.

References: [618], [893], [1101], [1323], [1456]. The proofs of 1.8.2 and 1.7.3 were communicated to me orally by F. Combes.

1.9. Addenda

1.9.1. Let A be a C^*-algebra. If the norm and all the algebraic operations of A are retained with the exception of $(\lambda,x)\to\lambda x$ which is changed to $(\lambda,x)\to\bar{\lambda}x$, a *$C^*$-algebra $\bar{A}$* called the conjugate of A is obtained.

1.9.2. Let A be a C^*-algebra. If $x\in A$ is normal, then $\|x\|=\rho(x)$. (Use 1.4.1) [1323].

1.9.3. Let A be a C^*-algebra.

(a) If every maximal commutative sub-C^*-algebra of A has an identity, so does A.

(b) If every maximal commutative sub-C^*-algebra of A is finite dimensional, so is A. [1159].

1.9.4. Let A be a C^*-algebra. If the conditions $x\in A^+$, $y\in A^+$, $x\geq y$ imply $x^2\geq y^2$, then A is commutative. [1160].

*1.9.5. Let A be a unital Banach algebra endowed with an involution such that $\|x^*x\| = \|x^*\| \cdot \|x\|$ for each $x \in A$. Then A is a C^*-algebra. [633], [1187].

*1.9.6. There exist unital C^*-algebras A of dimension >1, whose only closed two-sided ideals are 0 and A, having no projections other than 0 and 1. [1975], [2037].

1.9.7. Let A be a unital C^*-algebra, and x an element of A which does not have a left inverse. Then x^*x is not invertible in the sub-C^*-algebra B generated by 1 and x^*x, so there exist $y_1, y_2, \ldots \in B$ such that $\|y_n\| = 1$ and $\|y_n x^*x\| \to 0$ (use 1.4.1). Hence if $x \in A$ is not invertible, x is a topological divisor of zero. [1323].

1.9.8. Let A be a unital C^*-algebra, N the set of normal elements of A, $x \in N$ and V a neighbourhood of 0 in $\mathbf{C}$. There exists a neighbourhood U of x in N such that for every $y \in U$, $\operatorname{Sp} y \subset \operatorname{Sp} x + V$ and $\operatorname{Sp} x \subset \operatorname{Sp} y + V$. [1323].

*1.9.9. Let A be a unital C^*-algebra, H a Hilbert space and ϕ a linear map of A into $\mathscr{L}(H)$ such that $\phi(A^+) \subset \mathscr{L}(H)^+$. Then, for every hermitian x in A, we have $\phi(x^2) \geqslant \phi(x)^2$. [839].

*1.9.10. Let A and B be unital C^*-algebras.

(a) Let ρ: $A \to B$ be a bijective linear map such that $\rho(x^*) = \rho(x)^*$ and $\rho(x^n) = \rho(x)^n$ for x hermitian and n an integer >0. Then $\rho(A^+) = B^+$, ρ is isometric, ρ maps a pair of commuting elements into a pair of commuting elements, and $\rho(1) = 1$.

(b) Let σ: $A \to B$ be an isometric bijective linear map. Then σ is the composition of a map having the properties of (a) and multiplication on the left by the element $\sigma(1)$ of B.

(c) Let τ: $A \to B$ be an isometric linear map such that $\tau(1) = 1$. Then $\tau(x^*) = \tau(x)^*$ for each $x \in A$. [179], [837].

*1.9.11. Let A be a C^*-algebra.

(a) Every derivation of A is continuous.

(b) Let D be a derivation of A and x a normal element of A. If $x(Dx) = (Dx)x$, then $Dx = 0$. In particular, the only derivation of a commutative C^*-algebra is the zero operator. It follows from this that a commutative closed two-sided ideal in a C^*-algebra is central.

(c) Let D be a derivation of A and x a normal element of A such that $Dx = 0$. Then $Dx^* = 0$. [903], [1402], [1403].

(d) Let D be a derivation of A and I a closed two-sided ideal of A. Then $D(I) \subset I$ (use 1.5.8).

(e) Let H be an infinite-dimensional Hilbert space, A the C^*-algebra of all compact operators in H and x an element of $\mathscr{L}(H)$ which does not belong to $A + \mathbf{C} \cdot 1$. Then the derivation $y \to xy - yx$ of A is not inner. [530], [532], [535], [538], [1416], [1417], [1428], [1429], [1433].

1.9.12. Let A be a C^*-algebra.

(a) Let I and J be closed two-sided ideals of A. Then the product ideal IJ is equal to $I \cap J$. (If $x \in (I \cap J)^+$, then $x^{1/2} \in (I \cap J)^+$.) [508]. Further, $(I+J)^+ = I^+ + J^+$. [1239], [1515].

(b) If $I \cap J = 0$, the canonical map $I + J \to I \times J$ is a C^*-algebra isomorphism.

(c) Let (I_α) be a family of closed two-sided ideals of A whose intersection is I. Let ω_α (resp. ω) be the canonical morphism of A onto A/I_α (resp. A/I). For each $x \in A$, $\|\omega(x)\| = \sup_\alpha \|\omega_\alpha(x)\|$. [584].

1.9.13. Let A be a C^*-algebra. A C^*-semi-norm on A is a semi-norm N such that

$$N(x) \leq \|x\|, \qquad N(xy) \leq N(x)N(y), \qquad N(x^*x) = N(x)^2$$

for any $x, y \in A$. The set $\mathcal{N}(A)$ of all C^*-semi-norms on A is compact for the topology of pointwise convergence on A. If I is a closed two-sided ideal of A and $x \in A$, let $N_I(x)$ be the norm of the canonical image of x in A/I. Then $I \to N_I$ is a bijection of the set $\mathcal{I}(A)$ of all closed two-sided ideals of A onto $\mathcal{N}(A)$. If $I, J \in \mathcal{I}(A)$, we have $N_{I \cap J} = \sup(N_I, N_J)$ [use 1.9.12.b]. A C^*-semi-norm N is said to be extremal if N cannot be the upper bound of two C^*-semi-norms without being equal to one of them. For a C^*-semi-norm N to be non-zero and extremal, it is necessary and sufficient that I be prime. [In any algebra R, a two-sided ideal J is said to be prime if $J \neq R$ and if the relation $J'J'' \subset J$, where J', J'' are two-sided ideals of R, implies that $J' \subset J$ or $J'' \subset J$.] If A is separable, the set E of extremal C^*-semi-norms is a G_δ in $\mathcal{N}(A)$. [The set of $(N, N') \in \mathcal{N}(A) \times \mathcal{N}(A)$ such that $N \leq N'$ or $N' \leq N$ is compact. Its complement is a countable union of compact sets, and its image in $\mathcal{N}(A)$ under the continuous map $(N, N') \to \sup(N, N')$ is the complement of E.] [509], [584].

1.9.14. Let $(A_i)_{i \in I}$ be a family of C^*-algebras. Let A be the set of $x = (x_i) \in \prod_{i \in I} A_i$ such that for each $\epsilon > 0$, $\|x_i\| \leq \epsilon$ for all but a finite number of indices i. If we put $\|x\| = \sup\|x_i\|$, A becomes a C^*-algebra in a natural way, and this algebra is called the restricted product of the A_i.

1.9.15. The radical R of a C^*-algebra is zero. [If $x \in R$, then $1 + \lambda x^* x$ is invertible in $\tilde{A}$ for each $\lambda \in \mathbf{C}$, thus $\mathrm{Sp}'(x^* x) = 0$ and $x = 0$.] [1454].