. Let $n \in \mathbb{N}$,
e nth Taylor

on of $f^{(n)}(x_0)$

ablishing the

R.
how that, for

ted $a, b \in \mathbb{N}$
$= a/b$ in the
). Show that

7

Integration

In this chapter we present the theory of Riemann integration. While the process of integration had been developed much earlier in the seventeenth century by Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), it was Bernhard Riemann (1826–1866) who formulated the modern definition of the definite integral that is commonly used today. (We note in passing that some of the ideas of integral calculus can even be traced back to Archimedes in the third century B.C.) Subsequent to Riemann's work, other more general approaches to integration were developed, most notably by T. J. Stieltjes (1856–1894) and H. Lebesgue (1875–1941), but we shall not cover them in this book.

Since the reader has already seen many of the important applications of integration, we shall concentrate on a rigorous development of the theory. We begin by defining the Riemann integral in terms of upper and lower sums. In Section 7.2 we identify two classes of functions that are integrable and then derive several related algebraic properties. The fundamental theorem of calculus is included in Section 7.3, as is a brief discussion of improper integrals.

Section 7.1 THE RIEMANN INTEGRAL

7.1.1 DEFINITION Let $[a, b]$ be an interval in $\mathbb{R}$. A **partition** P of $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If P and Q are two partitions of $[a, b]$ with $P \subseteq Q$, then Q is called a **refinement** of P.

7.1.2 DEFINITION Suppose that f is a bounded function defined on $[a,b]$ and that $P = \{x_0, \ldots, x_n\}$ is a partition of $[a,b]$. For each $i = 1, \ldots, n$ we let

$$M_i(f) = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$m_i(f) = \inf \{f(x) : x \in [x_{i-1}, x_i]\}.$$

When only one function is under consideration, we may abbreviate these to M_i and m_i, respectively. Letting $\Delta x_i = x_i - x_{i-1}$ $(i = 1, \ldots, n)$, we define the **upper sum** of f with respect to P to be

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and the **lower sum** of f with respect to P to be

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

[Sometimes $U(f,P)$ and $L(f,P)$ are called the upper and lower Darboux sums in honor of Gaston Darboux (1842–1917), who first developed this approach to the Riemann integral.]

Since we are assuming that f is a bounded function on $[a,b]$, there exist numbers m and M such that $m \le f(x) \le M$ for all $x \in [a,b]$. Thus for any partition P of $[a,b]$ we have

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

This implies that the upper and lower sums for f form a bounded set, and it guarantees the existence of the following upper and lower integrals of f.

7.1.3 DEFINITION Let f be a bounded function defined on $[a,b]$. Then

$$U(f) = \inf \{U(f,P) : P \text{ is a partition of } [a,b]\}$$

is called the **upper integral** of f on $[a,b]$. Similarly,

$$L(f) = \sup \{L(f,P) : P \text{ is a partition of } [a,b]\}$$

is called the **lower integral** of f on $[a,b]$. If these upper and lower integrals are equal, then we say that f is **Riemann integrable** on $[a,b]$, and we denote their common value by $\int_a^b f$ or by $\int_a^b f(x)\,dx$. That is, if $L(f) = U(f)$, then

$$\int_a^b f = \int_a^b f(x)\,dx = L(f) = U(f)$$

is the **Riemann integral** of f on $[a,b]$.

Since the Riemann integral is the only kind of integral we shall deal with in this book, it will often be convenient to drop the reference to Riemann and simply refer to a function f as being **integrable** on $[a,b]$ and call $\int_a^b f$ the **integral** of f on $[a, b]$.

When the function f is nonnegative on $[a,b]$, we may interpret $\int_a^b f$ intuitively as the area under the graph of f between a and b. (We say "intuitively," since we shall not give a precise definition of "area" under the graph.) Each lower sum $L(f,P)$ represents the area of a union of rectangles with base Δx_i and height m_i. (See Figure 1.) Similarly, each upper sum $U(f,P)$ represents the area of a union of rectangles with base Δx_i and height M_i. (See Figure 2.) For any partition P, the area A under the graph of f is seen to satisfy $L(f,P) \le A \le U(f,P)$. But when $\int_a^b f$ exists, it is the *unique* number that lies between $L(f,P)$ and $U(f,P)$ for all partitions P. Thus $\int_a^b f$ corresponds to our intuitive notion of the area under the graph of f between a and b.

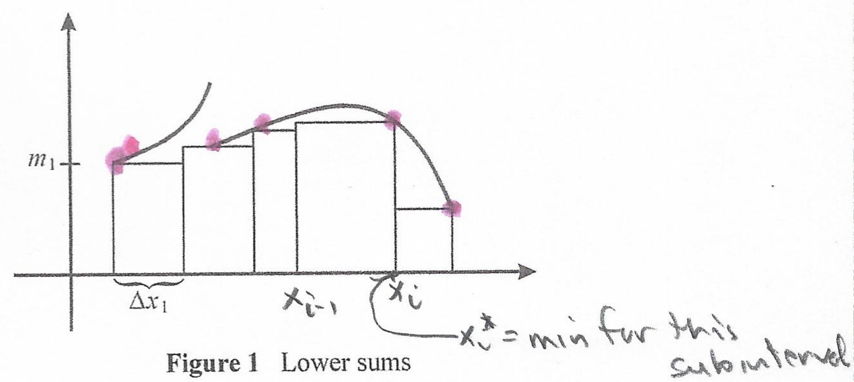


Figure 1 Lower sums

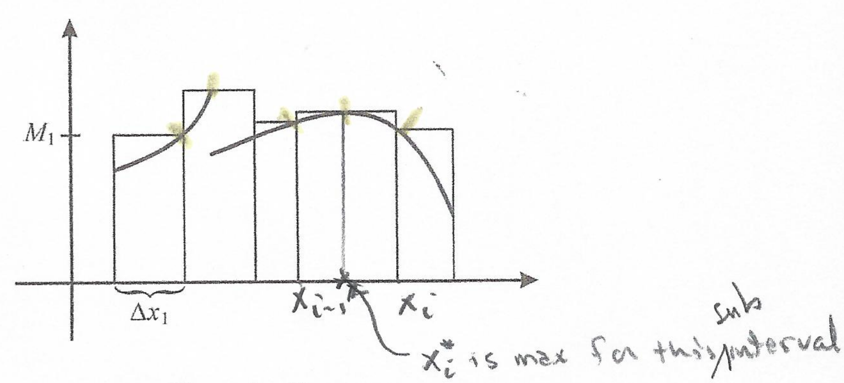
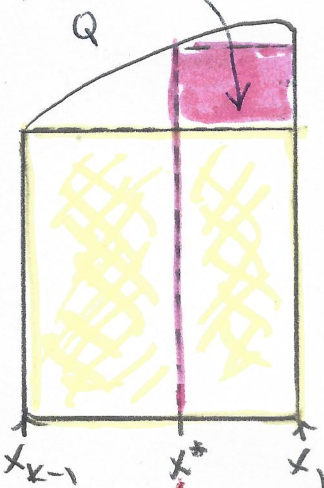


Figure 2 Upper sums

7.1.4 THEOREM Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ and Q is a refinement of P, then $P \subset Q$

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

increase going from P to Q



min for $[x_{k-1}, x_k]$ and min for $[x_{k-1}, x^*]$

min for $[x^*, x_k]$

Proof: The middle inequality follows directly from the definitions of $L(f,Q)$ and $U(f,Q)$. To prove $L(f,P) \le L(f,Q)$, we suppose that $P = \{x_0, x_1, \ldots, x_n\}$ and consider the partition P^* formed by joining some point, say x^*, to P, where $x_{k-1} < x^* < x_k$ for some $k = 1, \ldots, n$. Let

$$t_1 = \inf\{f(x) : x \in [x_{k-1}, x^*]\},$$
$$t_2 = \inf\{f(x) : x \in [x^*, x_k]\}.$$

Then $t_1 \ge m_k$ and $t_2 \ge m_k$, where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$, as usual.

Now the terms in $L(f,P^*)$ and $L(f,P)$ are all the same except those over the subinterval $[x_{k-1}, x_k]$. Thus we have

$$\begin{aligned} L(f,P^*) - L(f,P) &= [t_1(x^* - x_{k-1}) + t_2(x_k - x^*)] - [m_k(x_k - x_{k-1})] \\ &= (t_1 - m_k)(x^* - x_{k-1}) + (t_2 - m_k)(x_k - x^*). \end{aligned}$$

This final sum is nonnegative since all of the terms are nonnegative. Thus $L(f,P^*) - L(f,P) \ge 0$. Geometrically, adding the point x^* between x_{k-1} and x_k provides a closer approximation to the curve $y = f(x)$ and the lower sum (the area in the inscribed rectangles) is increased. (See Figure 3.)

Finally, if the partition Q contains r more points than P, we apply the argument above r times to obtain $L(f,P) \le L(f,Q)$.

The proof that $U(f,Q) \le U(f,P)$ is similar. ♦

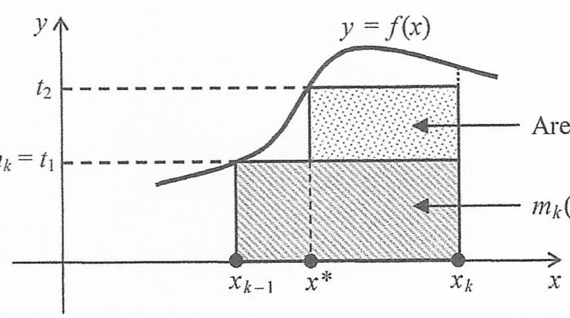


Figure 3 The lower sum increases.

7.1.5 PRACTICE Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$, prove that $L(f,P) \le U(f,Q)$. (*Hint*: Consider the partition $P \cup Q$ and notice that $P \cup Q$ is a refinement of both P and Q.)

$$L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q)$$

7.1.6 THEO

7.1.7 EXAM

7.1.6 THEOREM Let f be a bounded function on $[a, b]$. Then $L(f) \leq U(f)$.

Proof: If P and Q are partitions of $[a, b]$, then by Practice 7.1.5 we have $L(f, P) \leq U(f, Q)$. Thus $U(f, Q)$ is an upper bound for the set

$$S = \{L(f, P): P \text{ is a partition of } [a, b]\}.$$

It follows that $U(f, Q)$ is at least as large as $\sup S = L(f)$. That is, $L(f) \leq U(f, Q)$ for each partition Q of $[a, b]$. But then

$$L(f) \leq \inf\{U(f, Q): Q \text{ is a partition of } [a, b]\} = U(f). \ \blacklozenge$$

7.1.7 EXAMPLE Let us illustrate upper and lower sums by using them to evaluate $\int_0^1 x^2\,dx$. For each $n \in \mathbb{N}$, consider the partition

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

in which $\Delta x_i = 1/n$ for each $i = 1, 2, \ldots, n$. Since $f(x) = x^2$ is an increasing function on $[0, 1]$, on any subinterval $[(i-1)/n, i/n]$ we have

$$M_i = \sup\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^2$$

and

$$m_i = \inf\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = f\left(\frac{i-1}{n}\right) = \left(\frac{i-1}{n}\right)^2.$$

Thus

$$U(f, P_n) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2)$$

$$= \frac{1}{n^3}\left[\frac{1}{6}(n)(n+1)(2n+1)\right] = \frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{2n}\right),$$

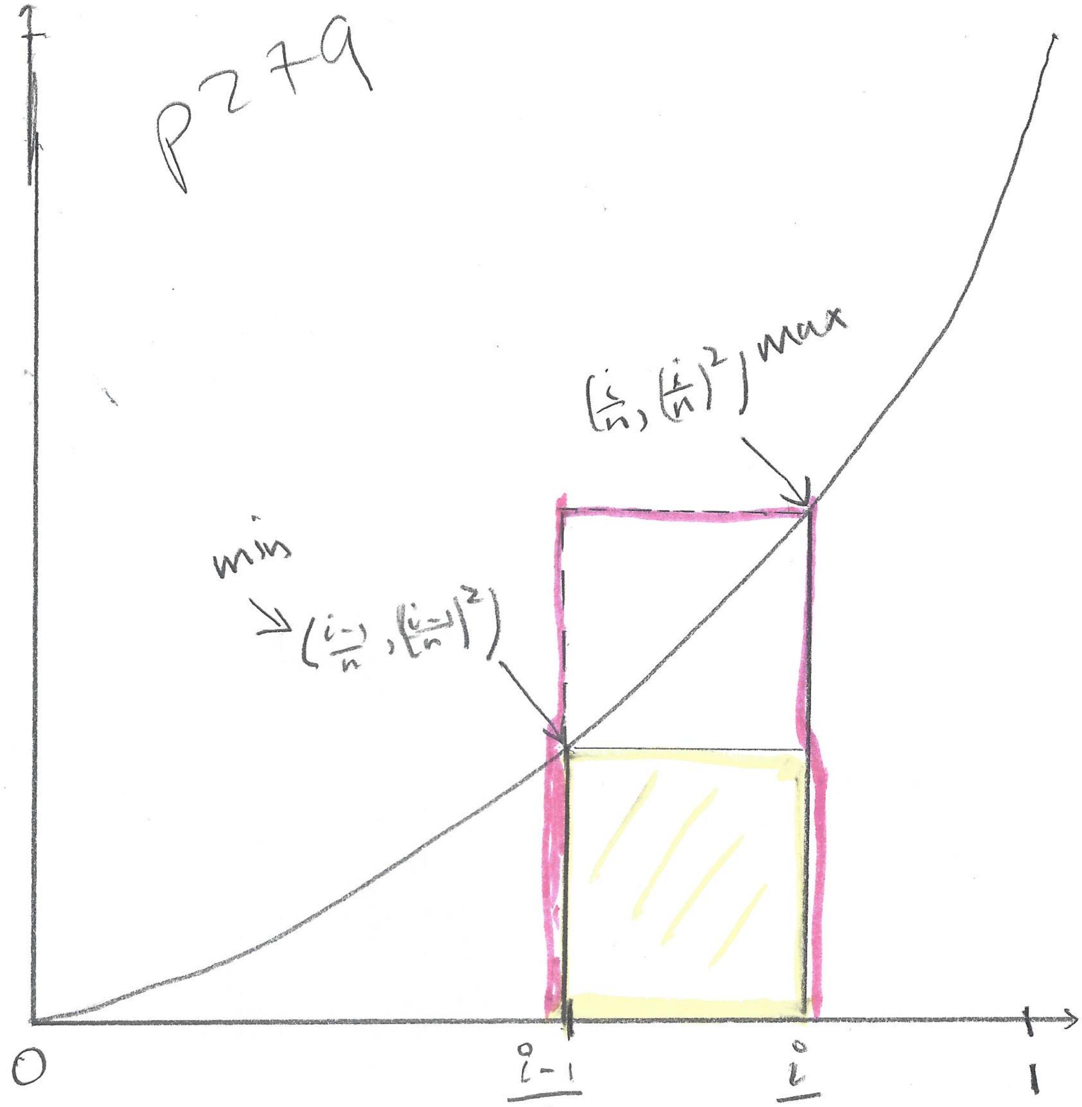
Prove by induction

where we have used the formula $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ from Exercise 3.1.3. Similarly,

$$L(f, P_n) = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}[0^2 + 1^2 + \cdots + (n-1)^2]$$

$$= \frac{1}{n^3}\left[\frac{1}{6}(n-1)(n)(2n-1)\right] = \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{2n}\right).$$

Since $\lim_{n\to\infty} U(f, P_n) = 1/3$ and $\lim_{n\to\infty} L(f, P_n) = 1/3$, we must have $U(f) \leq 1/3$ and $L(f) \geq 1/3$. But since $L(f) \leq U(f)$ by Theorem 7.1.6, this means that $L(f) = U(f) = 1/3$, so that $\int_0^1 x^2\,dx = 1/3$.



$$\Delta x_i = \frac{i}{n} - \frac{(i-1)}{n} = \frac{1}{n}$$

$$U(f, P_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right) = \frac{1}{n^3}\left[\frac{1}{6} n(n+1)(2n+1)\right]$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{2n}\right)$$

$$L(f, P_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\left[0^2 + 1^2 + \dots + (n-1)^2\right]$$

$$= \frac{1}{n^3}\left[\frac{1}{6}(n-1)\, n\, (2n-1)\right]$$

$$= \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{2n}\right)$$

$$U(f, P_n) \overset{\text{as } n\to\infty}{\downarrow} \frac{1}{3} \uparrow L(f, P_n)$$

$$\therefore \int_0^1 x^2\, dx = \frac{1}{3}$$

Using upper and lower sums as in Example 7.1.7 is a difficult way to calculate the value of an integral, even for a simple function like $f(x) = x^2$. Fortunately, the fundamental theorem of calculus (Theorem 7.3.5) will provide a much easier approach. Comparing upper and lower sums can, however, be useful in showing that a given function is *not* integrable.

7.1.8 EXAMPLE As a contrast to Example 7.1.7, we consider the function $g: [0,2] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,2]$. Since each subinterval $[x_{i-1}, x_i]$ contains both rational and irrational numbers, we have $M_i = 1$ and $m_i = 0$ for all $i = 1, \ldots, n$. Thus

$$U(g,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 2 \quad \text{and} \quad L(g,P) = \sum_{i=1}^{n} m_i \Delta x_i = 0.$$

It follows that $U(g) = 2$ and $L(g) = 0$. Since the upper and lower integrals of g on $[0,2]$ are not equal, we conclude that g is *not* integrable on $[0,2]$.

Since not every function is integrable, we are faced with the problem of determining when the integral of a function exists. In Section 7.2 we identify two large classes of functions that are integrable. Our next theorem will be very useful to us in that task.

7.1.9 THEOREM Let f be a bounded function on $[a,b]$. Then f is integrable iff for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof: Suppose that f is integrable, so that $L(f) = U(f)$. Given $\varepsilon > 0$, there exists a partition P_1 of $[a,b]$ such that

$$L(f,P_1) > L(f) - \frac{\varepsilon}{2}.$$

[This follows from the definition of $L(f)$ as a supremum.] Similarly, there exists a partition P_2 of $[a,b]$ such that

$$U(f,P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ and apply Theorem 7.1.4 to obtain

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\ &< \left[U(f) + \frac{\varepsilon}{2}\right] - \left[L(f) - \frac{\varepsilon}{2}\right] \\ &= U(f) - L(f) + \varepsilon = \varepsilon. \end{aligned}$$

Conversely, given $\varepsilon > 0$, suppose that there exists a partition P of $[a,b]$ such that $U(f,P) < L(f,P) + \varepsilon$. Then we have

$$U(f) \le U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we must have $U(f) \le L(f)$. But then Theorem 7.1.6 implies that $L(f) = U(f)$, so that f is integrable. ♦

Review of Key Terms in Section 7.1

Partition	Lower sum	Lower integral
Refinement	Upper integral	Riemann integral
Upper sum		

ANSWER TO PRACTICE PROBLEM

7.1.5 Since $P \cup Q$ is a refinement of both P and Q, Theorem 7.1.4 implies that $L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q)$.

7.1 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Let f be a bounded function defined on the interval $[a,b]$. Mark each statement True or False. Justify each answer.

(a) The upper and lower sums for f always form a bounded set.

(b) If P and Q are partitions of $[a,b]$, then $P \cup Q$ is a refinement of both P and Q.

(c) f is Riemann integrable iff its upper and lower sums are equal.

2. Let f be a bounded function defined on the interval $[a, b]$. Mark each statement True or False. Justify each answer.

(a) When $\int_a^b f$ exists, it is the unique number that lies between $L(f, P)$ and $U(f, P)$ for all partitions P of $[a, b]$.

(b) If P and Q are partitions of $[a, b]$, then $L(f, P) \le U(f, Q)$.

(c) If f is integrable on $[a, b]$, then given any $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $L(f, P) > U(f) - \varepsilon$.

3. Let $f(x) = x^2$ on $[0.5, 3]$. ☆

(a) Find $L(f, P)$ and $U(f, P)$ when $P = \{0.5, 1, 2, 3\}$.

(b) Find $L(f, P)$ and $U(f, P)$ when $P = \{0.5, 1, 1.5, 2, 2.5, 3\}$.

(c) Use calculus to evaluate $\int_{0.5}^3 x^2\,dx$.

4. Let $f(x) = 1/x$ on $[1, 3]$.

(a) Find $L(f, P)$ and $U(f, P)$ when $P = \{1, 2, 3\}$.

(b) Find $L(f, P)$ and $U(f, P)$ when $P = \{1, 1.5, 2, 2.5, 3\}$.

(c) Use calculus to evaluate $\int_1^3 1/x\,dx$.

5. Suppose that $f(x) = x$ for all $x \in [0, b]$. Show that f is integrable and that $\int_0^b f(x)\,dx = b^2/2$. ☆

6. Suppose that $f(x) = c$ for all $x \in [a, b]$. Show that f is integrable and that $\int_a^b f(x)\,dx = c(b - a)$.

7. Let $f(x) = x^3$ for $x \in [0, 1]$. Given $n \in \mathbb{N}$, let P_n be the partition of $[0, 1]$ defined in Example 7.1.7.

(a) Find $L(f, P_n)$ and $U(f, P_n)$. ☆

(b) Find $L(f)$ and $U(f)$.

8. Give an example of a function $f\colon [0, 1] \to \mathbb{R}$ that is not integrable on $[0, 1]$, but f^2 is integrable on $[0, 1]$.

9. Prove or give a counterexample: If f and g are integrable on $[a, b]$ and h is a function such that $f(x) \le h(x) \le g(x)$ for all $x \in [a, b]$, then h is integrable on $[a, b]$.

10. Define $f\colon [0, 1] \to \mathbb{R}$ by $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational.

(a) Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$, show that $U(f, P) > 1/2$. [*Hint*: Note that $M_i = x_i$ and that $x_i > (x_i + x_{i-1})/2$ since $x_i > x_{i-1}$.]

(b) For $n \in \mathbb{N}$, let P_n be the partition of $[0, 1]$ defined in Example 7.1.7. Show that $\lim_{n\to\infty} U(f, P_n) = 1/2$.

(c) Show that $U(f) = 1/2$ and $L(f) = 0$, so that f is not integrable on $[0, 1]$.

]. Mark each

en $L(f, P)$ and

ts a partition P

rable and that

rable and that

ition of $[0, 1]$

ble on $[0, 1]$,

$, b]$ and h is a
integrable on

is irrational.
at $U(f, P) >$
$x_i > x_{i-1}$.]
ample 7.1.7.

le on $[0, 1]$.

11. Let f be a bounded function on $[a, b]$. Suppose that there exists a sequence (P_n) of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

(a) Prove that f is integrable.

(b) Prove that $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

12. Let f be a bounded function on $[a, b]$ and suppose that $f(x) \geq 0$ for all $x \in [a, b]$. Prove that $L(f) \geq 0$.

***13.** Let f be continuous on $[a, b]$ and suppose that $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in [a, b]$. ☆

14. Let f and g be bounded functions on $[a, b]$.

(a) Prove that $U(f + g) \leq U(f) + U(g)$.

(b) Find an example to show that a strict inequality may hold in part (a).

15. Suppose that f is integrable on $[a, b]$ and that $[c, d] \subseteq [a, b]$. Prove that f is integrable on $[c, d]$.

***16.** Let $S = \{s_1, s_2, \ldots, s_k\}$ be a finite subset of $[a, b]$. Suppose that f is a bounded function on $[a, b]$ such that $f(x) = 0$ if $x \notin S$. Show that f is integrable and that $\int_a^b f = 0$.

17. Suppose that f is integrable on $[a, b]$. Use Theorem 7.1.9 to prove that f^2 is integrable on $[a, b]$.

18. Let f be a bounded function on $[a, b]$. In this exercise we identify an alternative way of obtaining the Riemann integral of f by using Riemann sums. (This approach is often used in beginning calculus texts.) Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, we define the **mesh** of P by

$$\text{mesh}\,(P) = \max\,\{\Delta x_i : i = 1, \ldots, n\}.$$

A **Riemann sum** of f associated with P is a sum of the form

$$\sum_{i=1}^{n} f(t_i)\Delta x_i,$$

where $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. Notice that the choice of t_i in $[x_{i-1}, x_i]$ is arbitrary, so that there are infinitely many Riemann sums associated with each partition.

(a) Prove that f is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $U(f, P) - L(f, P) < \varepsilon$ whenever P is a partition of $[a, b]$ with mesh $(P) < \delta$.

(b) Prove that f is integrable on $[a, b]$ iff there exists a number r such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|S - r| < \varepsilon$ for every Riemann sum S of f associated with a partition P such that mesh $(P) < \delta$.