6

Differentiation

Having developed our skill at working with limits, we now apply this understanding to the important process of differentiation. Most of the topics covered here will be at least somewhat familiar to the reader from the standard calculus course. In that earlier course a good deal of time was spent on the applications of derivatives in physics, geometry, economics, and the like. By way of contrast, the focus of this chapter will be on the theoretical aspects of differentiation that are often treated more superficially in the introductory course.

After establishing the basic properties of the derivative in Section 6.1, we prove the mean value theorem and develop some of its consequences in Section 6.2. In Section 6.3 we examine indeterminate forms and derive l'Hospital's rule for evaluating them. Finally, in Section 6.4 we give a brief discussion of Taylor's theorem.

Section 6.1 THE DERIVATIVE

6.1.1 DEFINITION Let f be a real-valued function defined on an interval I containing the point c. (We allow the possibility that c is an endpoint of I.) We say that f is **differentiable at** c (or has a **derivative at** c) if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$ so that

$$\left.\frac{df}{dx}\right|_{x=c} = f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

whenever the limit exists and is finite. If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be differentiable on S, and the function $f' : S \to \mathbb{R}$ is called the derivative of f on S.

6.1.2 EXAMPLE Let $f(x) = x^2$ for each $x \in \mathbb{R}$. Then for any $c \in \mathbb{R}$ we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c}$$
$$= \lim_{x \to c} (x + c) = 2c.$$

It is customary to regard f' as a function of x when f is a function of x, so we have $f'(x) = 2x$ for all $x \in \mathbb{R}$.

Geometrically, the difference quotient $\dfrac{f(x) - f(c)}{x - c}$ represents the slope of the secant line through the points $(c, f(c))$ and $(x, f(x))$. For example, when $c = 1/2$ and $x = 2$ we find the difference quotient is

$$\frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = \frac{4 - \frac{1}{4}}{2 - \frac{1}{2}} = \frac{5}{2}$$

As x approaches c, this ratio approaches the slope of the tangent line at c, in this case a slope of 1. (See Figure 1.)

Applying the sequential criterion for limits (Theorem 5.1.8)*, we obtain the following sequential condition for derivatives. The sequential approach is often useful when trying to show that a given function is *not* differentiable at a particular point.

6.1.3 THEOREM Let I be an interval containing the point c and suppose that $f: I \to \mathbb{R}$. Then f is differentiable at c iff, for every sequence (x_n) in I that converges to c with $x_n \neq c$ for all n, the sequence

$$\left(\frac{f(x_n) - f(c)}{x_n - c}\right)$$

converges. Furthermore, if f is differentiable at c, then the sequence of quotients above will converge to $f'(c)$.

* In Theorem 5.1.8 replace $\lim_{x \to c} f(x) = L$

by $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L = f'(c)$

6.1.4 EXA

$\lim_{x \to c} f(x) = L$ (A) $\Leftrightarrow$ (B) sequential convergence

5.1.8 THEOREM Let $f: D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L.

B ⇒ A

⇔ ~A ⇒ ~B

Proof: Suppose that $\lim_{x \to c} f(x) = L$ and let (s_n) be a sequence in D that converges to c with $s_n \neq c$ for all n. We must show that $\lim_{n \to \infty} f(s_n) = L$. Now, given any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$. Furthermore, since $s_n \to c$, there exists a natural number N such that $n \geq N$ implies that $|s_n - c| < \delta$. Thus for $n \geq N$ we have $0 < |s_n - c| < \delta_\varepsilon$ and $s_n \in D$, so that $|f(s_n) - L| < \varepsilon$. Hence $\lim_{n \to \infty} f(s_n) = L$.

A ⇒ B

N_{δ_ε}

Conversely, suppose that L is not a limit of f at c. We must find a sequence (s_n) in D that converges to c with each $s_n \neq c$, and such that $(f(s_n))$ does not converge to L. Since L is not a limit of f at c, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in D$ with $0 < |x - c| < \delta$ such that $|f(x) - L| \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists $s_n \in D$ with $0 < |s_n - c| < 1/n$ such that $|f(s_n) - L| \geq \varepsilon$. Now the sequence (s_n) converges to c with $s_n \neq c$ for all n, but $(f(s_n))$ cannot converge to L. ♦

Using Theorem 5.1.8 and our earlier results on sequences, we conclude that the limit of a function is unique.

5.1.9 COROLLARY If $f: D \to \mathbb{R}$ and if c is an accumulation point of D, then f can have only one limit at c.

$\lim_{x \to c} f(x) = L_1$, $\lim_{x \to c} f(x) = L_2$

$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon$ if $0 < |x - c| < \min(\delta_1, \delta_2)$

$< \varepsilon/2$ if $0 < |x - c| < \delta_{\varepsilon/2}$; $< \varepsilon/2$ if $0 < |x - c| < \delta_{\varepsilon/2}$

Proof: Exercise 10. ♦

One very useful application of the sequential criterion for the limit of a function is to show that a given limit does *not* exist.

5.1.10 THEOREM Let $f: D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following are equivalent:

Basically 5.1.8 ~A ⇔ ~B

$\lim_{x \to c} f(x) \neq L \ \forall L \in \mathbb{R}$

(a) f does not have a limit at c.

(b) There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is not convergent in $\mathbb{R}$.

Proof: Exercise 11. ♦

5.1.11 EXAMPLE Consider the function $f(x) = \sin(1/x)$ for $x > 0$. (See Figure 2.) Using Theorem 5.1.10, we can show that $\lim_{x \to 0} f(x)$ does not exist. Recall that for all $k \in \mathbb{N}$ we have

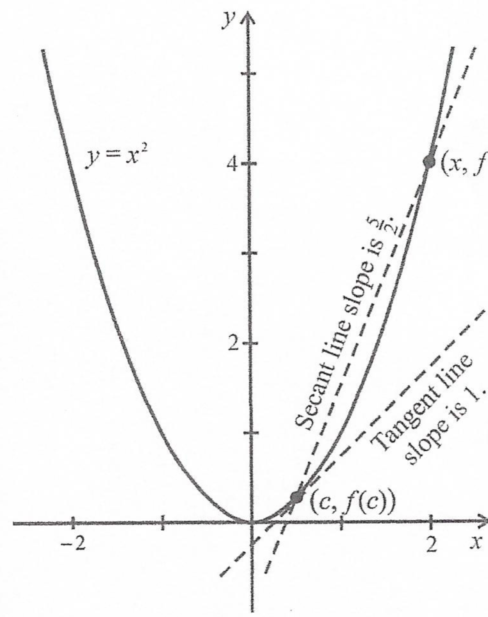


Figure 1 $f(x) = x^2$ with $c = 1/2$ and $x = 2$

6.1.4 EXAMPLE (a) Let $f(x) = |x|$ for each $x \in \mathbb{R}$, and let $x_n = (-1)^n/n$ for $n \in \mathbb{N}$. Then the sequence (x_n) converges to 0, but the corresponding sequence of quotients does not converge. (See Figure 2.)

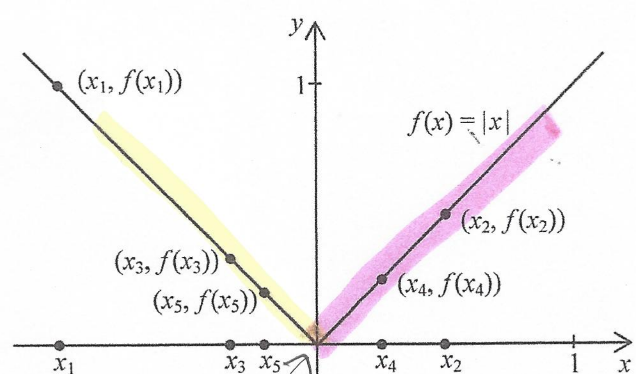


Figure 2 $f(x) = |x|$ and $x_n = (-1)^n/n$

Continuous but not differentiable at 0

Indeed, when n is even, $x_n = 1/n$ so that

$$\frac{f(x_n)-f(0)}{x_n-0} = \frac{1/n-0}{1/n-0} = 1.$$

But when n is odd we have $x_n = -1/n$, so that

$$\frac{f(x_n)-f(0)}{x_n-0} = \frac{1/n-0}{-1/n-0} = -1.$$

Since the two subsequences have different limits, the sequence

$$\left(\frac{f(x_n)-f(0)}{x_n-0}\right)$$

does not converge. Thus f is not differentiable at zero.

(b) Let $f(x) = 3x^2 + 1$ if $x < 1$ and $f(x) = 2x^3 + 2$ if $x \geq 1$. To see if f is differentiable at $x = 1$, we note that $f(1) = 4$ and look at the one-sided limits for the derivative at $x = 1$:

$$\begin{aligned}\lim_{x\to 1-}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1-}\frac{(3x^2+1)-4}{x-1} = \lim_{x\to 1-}\frac{3x^3-3}{x-1}\\ &= \lim_{x\to 1-} 3(x+1) = 3\cdot 2 = 6\end{aligned}$$

and
$$\begin{aligned}\lim_{x\to 1+}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1+}\frac{(2x^3+2)-4}{x-1} = \lim_{x\to 1+}\frac{2x^3-2}{x-1}\\ &= \lim_{x\to 1+} 2(x^2+x+1) = 2\cdot 3 = 6.\end{aligned}$$

Since these limits agree, we conclude that $f'(1)$ exists and is equal to 6.

$$\frac{f(x)-f(0)}{x-0} = \frac{x\sin\frac{1}{x}-0}{x-0} = \sin\frac{1}{x}$$

6.1.5 PRACTICE Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Determine whether or not f is differentiable at $x = 0$.

We see from Example 6.1.4(a) and Practice 6.1.5 that it is possible for a function to be continuous at a point without being differentiable at the point. On the other hand, it is easy to prove that if f is differentiable at a point, then it must also be continuous there.

6.1.6 THEOREM If $f\colon I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.

Proof: For every $x \in I$ with $x \neq c$, we have

$$f(x) = (x-c)\frac{f(x)-f(c)}{x-c} + f(c).$$

"differentiable" $\Rightarrow$ "continuous"

$\not\Leftarrow$

Since $f'(c)$ exists, we know that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \in \mathbb{R}.$$

Thus by Theorem 5.1.13 we obtain

$$\lim_{x \to c} f(x) = \left[\lim_{x \to c}(x - c)\right]\left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right] + \lim_{x \to c} f(c)$$
$$= 0 \cdot f'(c) + f(c) = f(c).$$

Hence Theorem 5.2.2(d) implies that f is continuous at c. ♦

We now present the useful (and familiar) rules for taking the derivative of sums, products, and quotients of functions.

6.1.7 THEOREM Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

(a) If $k \in \mathbb{R}$, then the function kf is differentiable at c and

$$(kf)'(c) = k \cdot f'(c).$$

(b) The function $f + g$ is differentiable at c and

$$(f + g)'(c) = f'(c) + g'(c).$$

Leibnitz' rule → (c) (Product Rule) The function fg is differentiable at c and

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c).$$

(d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

Proof: Parts (a) and (b) are left as exercises.

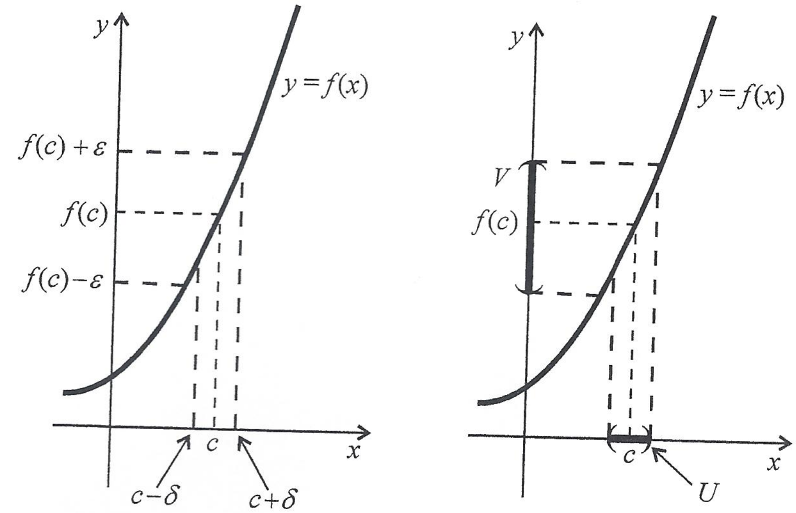
(c) For every $x \in I$ with $x \neq c$, we have

$$\frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} = \frac{(fg)(x) - (fg)(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}.$$

Theorem 6.1.6 implies that f is continuous at c, so $\lim_{x \to c} f(x) = f(c)$. Since f and g are differentiable at c, we conclude (using Theorem 5.1.13) that

$$(fg)'(c) = \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} = f(c)g'(c) + g(c)f'(c).$$

internal main idea $\lim_{x \to c} f(x) = f(\lim_{x \to c} x) = f(c)$



(a) ε–δ definition (b) using neighborhoods

Figure 1 Continuity of f at c

2 cases
c isolated element of D
c accumulation pt. of D

Notice that the definition of continuity at a point c requires c to be in D, but it does not require c to be an accumulation point of D. Thus the notion of continuity is slightly more general in application than the notion of a limit. Actually, the difference is not as significant as it might seem, since if c is an isolated point of D, then f is automatically continuous at c. (Recall that a point of D that is not an accumulation point of D is called an isolated point of D.) Indeed, if c is an isolated point of D, then there exists a $\delta > 0$ such that, if $|x - c| < \delta$ and $x \in D$, then $x = c$. Thus whenever $|x - c| < \delta$ and $x \in D$, we have

$p \Rightarrow q$

$\neg p \vee q$

$$|f(x) - f(c)| = 0 < \varepsilon$$

act jumps

for all $\varepsilon > 0$. Hence f is continuous at c.

if $\nexists$ such seq, we have $F \Rightarrow T$

5.2.2 THEOREM Let $f: D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:

$(p \Rightarrow q) \Leftrightarrow \neg p \vee q$

$\neg(p \Rightarrow q) \Leftrightarrow p \wedge \neg q$

(a) f is continuous at c.
(b) If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim_{n \to \infty} f(x_n) = f(c)$.
(c) For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.

p q

$x_n = c$ is allowed (isolated case)

Furthermore, if c is an accumulation point of D, then the above are all equivalent to

(d) f has a limit at c and $\lim_{x \to c} f(x) = f(c)$.

Main idea $\lim_{x \to c} f(x) = f(\lim_{x \to c} x) = f(c)$

(d) Since $g(c) \neq 0$ and g is continuous at c, there exists an interval $J \subseteq I$ with $c \in J$ such that $g(x) \neq 0$ for all $x \in J$. (See Exercise 5.2.13.) For all $x \in J$ with $x \neq c$, we have

$$\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c) = \frac{f(x)}{g(x)} - \frac{f(c)}{g(c)} = \frac{g(c)f(x) - f(c)g(x)}{g(x)g(c)}$$
$$= \frac{g(c)f(x) - g(c)f(c) + g(c)f(c) - f(c)g(x)}{g(x)g(c)}.$$

It follows that

$$\left(\frac{f}{g}\right)'(c) = \lim_{x \to c} \frac{(f/g)(x) - (f/g)(c)}{x - c}$$
$$= \lim_{x \to c} \left\{ \left[g(c)\frac{f(x) - f(c)}{x - c} - f(c)\frac{g(x) - g(c)}{x - c} \right] \left[\frac{1}{g(x)g(c)} \right] \right\}$$
$$= \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}. \quad \blacklozenge$$

6.1.8 EXAMPLE To illustrate the use of Theorem 6.1.7, let us show that for any $n \in \mathbb{N}$, if $f(x) = x^n$ for all $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. Our proof is by induction.

When $n = 1$ we have $f(x) = x$, so that

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t - x}{t - x} = \lim_{t \to x} 1 = 1 = 1x^0,$$

and the formula holds.† Now suppose that the formula holds for $n = k$. That is, if $f(x) = x^k$, then $f'(x) = kx^{k-1}$. We write the function $g(x) = x^{k+1}$ as the product of two functions and use the product rule of Theorem 6.1.7. Let $f(x) = x^k$ and $h(x) = x$. Then $g(x) = f(x)h(x)$, so that

$$g'(x) = f(x)h'(x) + h(x)f'(x)$$
$$= (x^k)(1) + (x)(kx^{k-1}) = (k+1)x^k.$$

Thus the formula holds for $n = k + 1$, and by induction we conclude that it holds for all $n \in \mathbb{N}$.

We also note that the formula holds for $n = 0$. That is, if $f(x) = 1$ for all x, then f is differentiable for all x and $f'(x) = 0$.

† When $n = 1$ so that $f(x) = x$, the derivative formula gives $f'(x) = 1x^0$. Clearly we want $f'(x)$ to be 1 when $x = 0$, and this requires us to interpret 0^0 to be 1. This is one of the reasons some mathematicians want to define $0^0 = 1$, at least in those cases where we are not dealing with limits such as $\lim_{x \to 0} f(x)^{g(x)}$, and $\lim f(x) = \lim g(x) = 0$. See Section 6.3.

* (d) Thm 6.1.10 will give us derivative of $\frac{1}{g} = g^{-1}$ as

$\frac{d\, g^{-1}(x)}{dx} = (-1)[g(x)]^{-2} g'(x)$ (chain rule) + (example 6.1.8) augmented (example 6.1.9.)

Thus $\left(\frac{f}{g}\right)' = (f g^{-1})' \overset{\text{by (c)}}{=} f' g^{-1} + f(-1) g^{-2} g'$

$= [f' g - f g'] g^{-2}$

p242

$n \in \mathbb{N} = \{1, 2, \ldots\}$

Example 6.1.8 If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

"alternate proof"   prove this first by induction

$$x^n - x_0^n = (x - x_0)\underbrace{(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + x_0^{n-1})}_{n \text{ terms}}$$

$$\lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim \frac{\cancel{(x - x_0)}}{\cancel{(x - x_0)}}(x^{n-1} + x^{n-2}x_0 + \ldots + x_0^{n-1})$$

$$= \lim_{x \to x_0}\left[x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + x\,x_0^{n-2} + x_0^{n-1}\right]$$

$$= \underbrace{x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \ldots x_0 x_0^{n-2} + x_0^{n-1}}_{n \text{ terms}}$$

$$= n x_0^{n-1}$$

p243

Example/practice 6.1.9 Same formula works for negative integer exponents, i.e.

Let $f(x) = \frac{1}{x^n} = x^{-n}$ $n \in \mathbb{N} = \{1, 2, \ldots\}$

Then $f'(x) = (-n)x^{-n-1}$ (Same formula as before)

Proof: $\frac{f(x) - f(x_0)}{x - x_0}$ becomes:

$$\frac{\frac{1}{x^n} - \frac{1}{x_0^n}}{x - x_0} = \frac{x_0^n - x^n}{x^n x_0^n (x - x_0)} =$$ by (previous algebra)

$$= \frac{\cancel{(x_0 - x)}(x_0^{n-1} + x_0^{n-2}x + x_0^{n-3}x^2 + \ldots + x_0 x^{n-2} + x^{n-1})}{x_0^n x^n (x - x_0)}$$

$$\Rightarrow \lim_{x \to x_0} \frac{(-1)}{x_0^n x^n} (x_0^{n-1} + x_0^{n-2}x + \cdots + x^{n-1})$$

$$= \frac{(-1)}{x_0^{2n}} (\underbrace{x_0^{n-1} + x_0^{n-1} + \cdots + x_0^{n-1}}_{n \text{ terms}})$$

$$= \frac{(-n)\, x_0^{n-1}}{x_0^{2n}} = -n x_0^{n-2n-1} = -n x^{-n-1}$$

Summary: If $f(x) = x^n$ $n \in \mathbb{Z}$

then $f'(x) = n x^{n-1}$

(where $\frac{dx^0}{dx} = 0$)

6.1.9 PRACTICE Let n be a negative integer and let $f(x) = x^n$ for $x \neq 0$. Note that $-n$ is positive and if $g(x) = x^{-n}$, then $f(x) = (1/g)(x)$. Use Example 6.1.8 and the quotient rule of Theorem 6.1.7 to show that $f'(x) = nx^{n-1}$.

In Section 5.2 we proved that the composition of two continuous functions is continuous. A similar result holds for the composition of differentiable functions, and it is known as the chain rule.

6.1.10 THEOREM (Chain Rule) Let I and J be intervals in $\mathbb{R}$, let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Proof: Following our usual approach, we write

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$

It would seem that by taking the limit of both sides as $x \to c$ we would obtain the desired result. The only problem is that $f(x) - f(c)$ may be zero even when $x - c \neq 0$. Thus the first factor in the right-hand side may have a zero denominator. To circumvent this problem, we note that

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c))$$

since g is differentiable at $f(c)$. Thus we define a new function $h: J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \dfrac{g(y) - g(f(c))}{y - f(c)}, & \text{if } y \neq f(c), \\ g'(f(c)), & \text{if } y = f(c), \end{cases}$$

and see that h is continuous at $f(c)$.

Now since f is differentiable at c, Theorem 6.1.6 implies that f is continuous at c. Hence $h \circ f$ is continuous at c by Theorem 5.2.12, so that

$$\lim_{x \to c} (h \circ f)(x) = h(f(c)) = g'(f(c)).$$

It follows from our definition of h that

$$g(y) - g(f(c)) = h(y)[y - f(c)], \qquad \text{for all } y \in J.$$

p243 CHAIN RULE

$$\lim_{x \to x_0} \frac{(g\circ f)(x) - (g\circ f)(x_0)}{x - x_0} =?\ (g\circ f)'(x_0)$$

Let's try $\dfrac{g[f(x)] - g[f(x_0)]}{f(x) - f(x_0)} \left(\dfrac{f(x) - f(x_0)}{x - x_0}\right)$

Now f is differentiable at x_0, so continuous

i.e. $f(x) \to f(x_0)$ as $x \to x_0$.

Taking limits if $f(x) - f(x_0) \neq 0$ near x_0

we can get

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = \lim_{f(x) \to f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$$

think in terms of sequences $f(x_n) \to f(x_0)$

$$= g'(f(x_0))$$

and $\displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

so $(g\circ f)'(x_0) = [g'(f(x_0))][f'(x_0)]$

But what if $f(x) = f(x_0)$ "near x_0"

then $\dfrac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = \dfrac{0}{0}$ so there is hope.

We define a new function "h" that agrees with $\dfrac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$ whenever possible

A

p243 Chain RULE (continued)

namely

$$h(y) = \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } y \neq f(x_0) \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$$

Now h is continuous, in particular, h is continuous at $f(x_0)$ since

$$\lim_{y \to f(x_0)} h(y) = \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0))$$

g is differentiable at $f(x_0)$

Recall f is continuous (at x_0 for example) because it is differentiable at x_0 so

$h \circ f$ is continuous (at x_0)

Thus $\lim_{x \to x_0} (h \circ f)(x) = h(f(x_0)) = g'(f(x_0))$

Now the definition of h implies

$g(y) - g(f(x_0)) = h(y)(y - f(x_0))$ for any y where g is defined. (like on Range f)
(let $y = f(x)$ for example)

Thus

$(g \circ f)(x) - (g \circ f)(x_0) = (h \circ f)(x)\,[f(x) - f(x_0)]$

So if x is in domain of f, i.e. $f(x)$ defined,
if $x \neq x_0$ we have from previous eqn

then $\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = (h \circ f)(x) \left[\frac{f(x) - f(x_0)}{x - x_0} \right]$

Now take $\lim_{x \to x_0}$ and we get

if limit exist

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} (h \circ f)(x) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g'(f(x_0))\, f'(x_0)$$

(c)

Thus, if $x \in I$, then $f(x) \in J$, so that

$$(g \circ f)(x) - (g \circ f)(c) = [(h \circ f)(x)][f(x) - f(c)].$$

But then for $x \in I$ with $x \neq c$, we have

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = [(h \circ f)(x)]\left[\frac{f(x) - f(c)}{x - c}\right].$$

Now we can take the limit of both sides as $x \to c$ to obtain

$$\begin{aligned}(g \circ f)'(c) &= \lim_{x \to c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} \\ &= \lim_{x \to c} [(h \circ f)(x)] \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= g'(f(c)) \cdot f'(c). \quad \blacklozenge\end{aligned}$$

6.1.11 EXAMPLE Let us return to the function defined in Practice 6.1.5. That is, $f(x) = x \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Using the fact that the derivative of $\sin x$ is $\cos x$ for all $x \in \mathbb{R}$, we compute the derivative of f at any point $x \neq 0$. Let $h(x) = \sin(1/x)$, so that $f(x) = xh(x)$. Then for $x \neq 0$ the chain rule gives us

$$h'(x) = \left[\cos\frac{1}{x}\right][-x^{-2}],$$

where we have differentiated $1/x = x^{-1}$ using the formula from Practice 6.1.9. Thus by the product rule of Theorem 6.1.7 we have

$$\begin{aligned}f'(x) &= xh'(x) + h(x) \\ &= -\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x}\end{aligned}$$

for all $x \neq 0$. We saw in Practice 6.1.5 that $f'(0)$ does not exist, so f is a (continuous) function that has a derivative at every real x except $x = 0$.[†]

Review of Key Terms in Section 6.1

Differentiable at c	Chain rule

[†] We remark in passing that there exist functions that are continuous for all real x but have a derivative at *no* points. We shall prove this in Chapter 9 (Theorem 9.2.9) as an application of uniform convergence.

ANSWERS TO PRACTICE PROBLEMS

6.1.5 For $x \neq 0$ we have

$$\frac{f(x)-f(0)}{x-0} = \frac{x \sin(1/x)}{x} = \sin\frac{1}{x}.$$

In Example 5.1.11 we showed that $\lim_{x\to 0} \sin(1/x)$ does not exist. Thus f is not differentiable at $x = 0$.

6.1.9 Since $-n \in \mathbb{N}$, the function $g(x) = x^{-n}$ is differentiable by Example 6.1.8 and $g'(x) = -nx^{-n-1}$. Using the quotient rule of Theorem 6.1.7, we have

$$\begin{aligned} f'(x) &= \left(\frac{1}{g}\right)'(x) = \frac{[g(x)][0]-[1][g'(x)]}{[g(x)]^2} \\ &= \frac{-(-n)x^{-n-1}}{(x^{-n})^2} = nx^{n-1}. \end{aligned}$$

6.1 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Let c be a point in the interval I and suppose $f: I \to \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) The derivative of f at c is defined by

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

wherever the limit exists.

(b) If f is continuous at c, then f is differentiable at c.

(c) If f is differentiable at c, then f is continuous at c.

2. Let c be a point in the interval I and suppose $f: I \to \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) If f is differentiable at c, then for any $k \in \mathbb{R}$, kf is differentiable at c.

(b) Suppose $g: I \to \mathbb{R}$. If f and g are differentiable at c, then $f + g$ is differentiable at c.

(c) Suppose $g: I \to \mathbb{R}$. If f and g are differentiable at c, then $g \circ f$ is differentiable at c.

3. Determine if each function is differentiable at $x = 1$. If it is, find the derivative. If not, explain why not.

   (a) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

   (b) $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

   (c) $f(x) = \begin{cases} 3x-2 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

4. Use Definition 6.1.1 to find the derivative of each function.

   (a) $f(x) = 3x + 5$ for $x \in \mathbb{R}$

   (b) $f(x) = x^3$ for $x \in \mathbb{R}$

   (c) $f(x) = \dfrac{1}{x}$ for $x \neq 0$

   (d) $f(x) = \sqrt{x}$ for $x > 0$

   (e) $f(x) = \dfrac{1}{\sqrt{x}}$ for $x > 0$

5. Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$.

   (a) Use Definition 6.1.1 to prove that $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$. ☆

   (b) Show that f is not differentiable at $x = 0$.

*6. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

   (a) Use the chain rule and the product rule to show that f is differentiable at each $c \neq 0$ and find $f'(c)$. (You may assume that the derivative of $\sin x$ is $\cos x$ for all $x \in \mathbb{R}$.)

   (b) Use Definition 6.1.1 to show that f is differentiable at $x = 0$ and find $f'(0)$.

   (c) Show that f' is not continuous at $x = 0$.

   (d) Let $g(x) = x^2$ if $x \leq 0$ and $g(x) = x^2 \sin(1/x)$ if $x > 0$. Determine whether or not g is differentiable at $x = 0$. If it is, find $g'(0)$.

7. Determine for which values of x each function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative.

   (a) $f(x) = |x - 1|$

   (b) $f(x) = |x^2 - 1|$ ☆

   (c) $f(x) = |x|$

   (d) $f(x) = x|x|$ ☆

*8. Let $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$.

   (a) Show that f is differentiable on $\mathbb{R}$.

   (b) Show that f' is not bounded on the interval $[-1, 1]$.

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9. Let $f(x) = x^2$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$.
(a) Show that f is differentiable at $x = 0$. ☆
(b) Find $f'(x)$ for all real x and sketch the graph of f'.
(c) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$? ☆

10. Complete the proof of parts (a) and (b) of Theorem 6.1.7.

11. Let $f(x) = x^2$ if x is rational and $f(x) = 0$ if x is irrational.
(a) Prove that f is continuous at exactly one point, namely at $x = 0$.
(b) Prove that f is differentiable at exactly one point, namely at $x = 0$.

12. Prove: If a polynomial $p(x)$ is divisible by $(x - a)^2$, then $p'(x)$ is divisible by $(x - a)$.

13. Let f, g, and h be real-valued functions that are differentiable on an interval I. Prove that the product function $fgh: I \to \mathbb{R}$ is differentiable on I and find $(fgh)'$. ☆

14. Let $f: I \to J$, $g: J \to K$, and $h: K \to \mathbb{R}$, where I, J, and K are intervals. Suppose that f is differentiable at $c \in I$, g is differentiable at $f(c)$, and h is differentiable at $g(f(c))$. Prove that $h \circ (g \circ f)$ is differentiable at c and find the derivative.

15. Suppose that $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are differentiable at $c \in I$ and that $g(c) \neq 0$.
(a) Use Exercise 4(c) and the chain rule [Theorem 6.1.10] to show that $(1/g)'(c) = -g'(c)/[g(c)]^2$.
(b) Use part (a) and the product rule [Theorem 6.1.7(c)] to derive the quotient rule [Theorem 6.1.7(d)].

16. Let I and J be intervals and suppose that the function $f: I \to J$ is twice differentiable on I. That is, the derivative f' exists and is itself differentiable on I. (We denote the derivative of f' by f''.) Suppose also that the function $g: J \to \mathbb{R}$ is twice differentiable on J. Prove that $g \circ f$ is twice differentiable on I and find $(g \circ f)''$.

17. Let $f: I \to \mathbb{R}$, where I is an open interval containing the point c, and let $k \in \mathbb{R}$. Prove the following.
(a) f is differentiable at c with $f'(c) = k$ iff $\lim_{h \to 0}[f(c+h) - f(c)]/h = k$.
*(b) If f is differentiable at c with $f'(c) = k$, then $\lim_{h \to 0} [f(c + h) - f(c - h)]/2h = k$.
(c) If f is differentiable at c with $f'(c) = k$, then $\lim_{n \to \infty} n[f(c + 1/n) - f(c)] = k$.
(d) Find counterexamples to show that the converses of parts (b) and (c) are not true.

18. A function $f: \mathbb{R} \to \mathbb{R}$ is called an **even** function if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. If $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, then f is called an **odd** function.
   (a) Prove that if f is a differentiable even function, then f' is an odd function.
   (b) Prove that if f is a differentiable odd function, then f' is an even function.

19. Suppose that $f'(0)$ exists and that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$.
   (a) Use Exercise 17(a) to prove that f' exists for all $x \in \mathbb{R}$. ☆
   (b) Find several different functions that satisfy the given conditions.

Section 6.2 THE MEAN VALUE THEOREM

The mean value theorem (also called the law of the mean) is one of the most important theoretical results in differential calculus. Its proof depends on the fact that a continuous function defined on a compact set assumes its maximum and minimum values (Corollary 5.3.3). In this section we establish the theorem and derive several of its corollaries. We begin with a preliminary result about maxima and minima that is also of interest in its own right.

6.2.1 THEOREM If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Proof: Suppose that f assumes its maximum at c. That is, $f(x) \le f(c)$ for all $x \in (a, b)$. Let (x_n) be a sequence converging to c such that $a < x_n < c$ for all n. (Since $a < c$ we may, for example, take $x_n = c - 1/n$ for n sufficiently large.) Then, since f is differentiable at c, Theorem 6.1.3 implies that the sequence

$$\left(\frac{f(x_n) - f(c)}{x_n - c}\right)$$

converges to $f'(c)$. But each term in this sequence of quotients is nonnegative, since $f(x_n) \le f(c)$ and $x_n < c$. Thus $f'(c) \ge 0$ by Corollary 4.2.5.

Similarly, let (y_n) be a sequence converging to c such that $c < y_n < b$ for all n. Then the terms of the sequence

$$\left(\frac{f(y_n) - f(c)}{y_n - c}\right)$$