$+ 6$ if x is
2. there

Suppose that

it c? Justify

it

$f - g|$. ☆

$\sqrt{f} : D \to \mathbb{R}$
$\bar{f}$ is contin-

. Prove that
$> \alpha$ for all

n $M > 0$ and

$f^{-1}(G)$.

a c' ed set

$f(x + y) =$
at $f(x) = kx$,

ery rational

$s_1 = c$ and

Section 5.3 PROPERTIES OF CONTINUOUS FUNCTIONS

In this section we develop a number of the important properties of continuous functions. One of these, the intermediate value property, is probably familiar to the reader from calculus, where it is usually presented without proof. Once again, we shall consider functions whose domain D is a nonempty subset of $\mathbb{R}$.

A function $f: D \to \mathbb{R}$ is said to be **bounded** if its range $f(D)$ is a bounded subset of $\mathbb{R}$. That is, f is bounded if there exists $M \in \mathbb{R}$ such that $|f(x)| \le M$ for all $x \in D$. Unfortunately, a continuous function may not be bounded even when its domain is bounded.

5.3.1 PRACTICE Let $D = (0, 1)$. Find a continuous function that is not bounded on D. $\frac{1}{x}$

If it happens, however, that the domain of a continuous function is both closed and bounded, then the function will be bounded. In fact, we can prove the following stronger result. Recall from the Heine–Borel theorem (3.5.5) that a subset of $\mathbb{R}$ is compact iff it is closed and bounded. The open-cover property of compact sets will be particularly useful in proving the theorem.

5.3.2 THEOREM Let D be a compact subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Proof: Let $\mathcal{G} = \{G_\alpha\}$ be an open cover of $f(D)$. We will show that $\mathcal{G}$ has a finite subcover. Since f is continuous on D, Theorem 5.2.14 implies that for each open set G_α in $\mathcal{G}$, there is an open set H_α such that $\underbrace{H_\alpha \cap D}_{\text{relatively open in } D} = f^{-1}(G_\alpha)$. Furthermore, since $f(D) \subseteq \bigcup G_\alpha$, it follows that

$$D \subseteq \bigcup f^{-1}(G_\alpha) \subseteq \bigcup H_\alpha. \qquad D \subset f^{-1}(f(D))$$

Thus the collection $\{H_\alpha\}$ is an open cover of the set D. Since D is compact, there exist finitely many sets $H_{\alpha_1}, \ldots, H_{\alpha_n}$ such that

$$D \subseteq H_{\alpha_1} \cup \cdots \cup H_{\alpha_n}.$$

But then

$$D \subseteq (H_{\alpha_1} \cap D) \cup \cdots \cup (H_{\alpha_n} \cap D) \quad ☆$$

and

$$f(D) \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

$f(f^{-1}(G_\alpha)) \subset G_\alpha$

Not all of G_α may be in the range of f

Thus $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ is a finite subcover of $\mathcal{G}$ for $f(D)$ and $f(D)$ is compact. ♦

☆ Use $f\left(\bigcup_{\alpha \in A} S_\alpha\right) = \bigcup_{\alpha \in A} f(S_\alpha)$

Note: $f\left(\bigcap_{\alpha \in A} S_\alpha\right) \subseteq \bigcap_{\alpha \in A} f(S_\alpha)$

Exercise: Find simple example when this is $\subset$ but NOT $=$

5.3.3 COROLLARY Let D be a compact subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in D$.

A nonempty subset of ℝ which is closed and bounded has a maximum and a minimum.

Proof: We know from Theorem 5.3.2 that $f(D)$ is compact. Thus (Lemma 3.5.4) $f(D)$ has both a minimum, say y_1, and a maximum, say y_2. Since $y_1, y_2 \in f(D)$, there exist $x_1, x_2 \in D$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. It follows that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in D$. ♦

$f(x) = x$

To apply Corollary 5.3.3, it is necessary that D be both closed and bounded. For example, the identity function on the unbounded set $[0, \infty)$ is continuous, but it certainly does not assume a maximum value. If D is not closed, then f may not attain its supremum and infimum on D even if $f(D)$ is bounded.

5.3.4 PRACTICE Let $D = (0, 1)$. Find a continuous function $f: D \to \mathbb{R}$ such that f is bounded on D but does not assume max and min values on D. $f(x) = x$

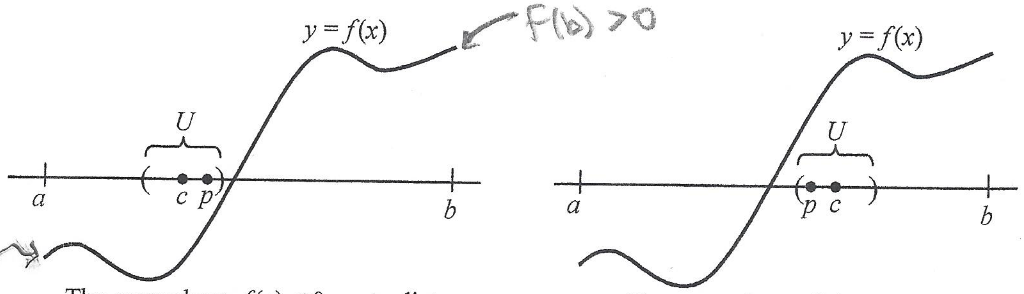
While the previous results apply to the continuous image of any compact subset of $\mathbb{R}$, if we require D to be a compact interval, then we obtain the following additional properties.

5.3.5 LEMMA Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(c) = 0$.

Proof: Our strategy is to let c be the largest x for which $f(x) \le 0$. More precisely, let $S = \{x \in [a, b]: f(x) \le 0\}$. Since $a \in S$, S is nonempty. Clearly S is bounded above by b, and so $c = \sup S$ exists as a real number in $[a, b]$. We claim that $f(c) = 0$. Indeed, suppose that $f(c) < 0$. Then there exists a neighborhood U of c such that $f(x) < 0$ for all $x \in U \cap [a, b]$. (See Exercise 5.2.13.) Now $c \ne b$ since $f(c) < 0 < f(b)$. Thus U contains a point p such that $c < p < b$. But $f(p) < 0$ since $p \in U$, so $p \in S$. This contradicts c being an upper bound for S. (See Figure 1.)

in HW #20

$f(b) > 0$



$f(a) < 0$

The case where $f(c) < 0$ contradicts c being an upper bound for S.

The case where $f(c) > 0$ contradicts c being the *least* upper bound for S.

Figure 1 Showing $f(c) = 0$

Similarly, if $f(c) > 0$, then there exists a neighborhood U of c such that $f(x) > 0$ for all $x \in U \cap [a, b]$. Now $c \neq a$ since $f(a) < 0 < f(c)$. Thus U contains a point p with $a < p < c$. Since $f(x) > 0$ for all x in U, no points of S are in $[p, c]$. This implies that p is an upper bound for S and contradicts c being the *least* upper bound for S. We conclude that $f(c) = 0$, as desired. Finally, since $f(a) < 0 < f(b)$ and $f(c) = 0$, it must be that $c \in (a, b)$. ♦

i.e. $a < c < b$

5.3.6 THEOREM (Intermediate Value Theorem) Suppose that $f: [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ [i.e., $f(a) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in (a, b)$ such that $f(c) = k$.

Proof: Let k be any number between $f(a)$ and $f(b)$. If $f(a) < f(b)$, then apply Lemma 5.3.5 to the continuous function $g: [a, b] \to \mathbb{R}$ given by $g(x) = f(x) - k$. Then $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0$. Thus there exists $c \in (a, b)$ such that $g(c) = f(c) - k = 0$. If $f(a) > f(b)$, a similar argument applies to the function $g(x) = k - f(x)$. ♦

The idea behind the intermediate value theorem is very simple. It says that the graph of f must cross any horizontal line between $y = f(a)$ and $y = f(b)$. (See Figure 2.) Intuitively, if the graph of f is below $y = k$ at a and above $y = k$ at b, then for f to be continuous on $[a, b]$, it must cross $y = k$ somewhere in between. Thus the graph of a continuous function can have no "jumps."

no matter how "small"

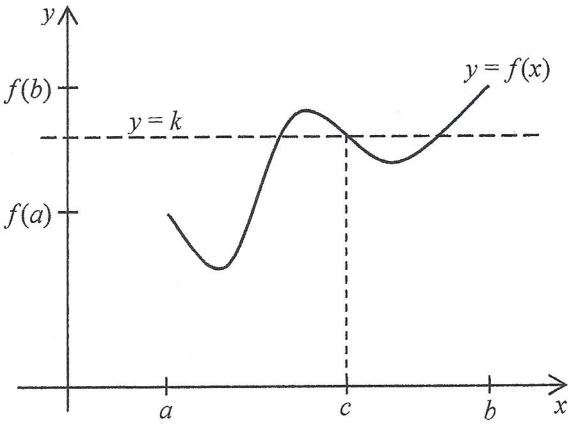
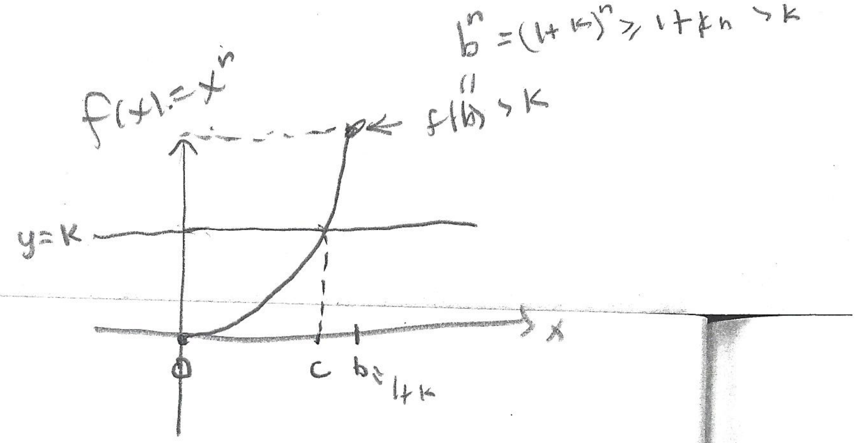


Figure 2 The intermediate value theorem



5.3.7 EXAMPLE Using the intermediate value theorem, we can show that every positive number has a positive nth root. Suppose that $k > 0$ and $n \in \mathbb{N}$. Let $f(x) = x^n$. Then $f(0) = 0 < k$. Furthermore, if $b = k + 1$, then

$$b^n = (1+k)^n \geq 1 + kn > k$$

by Bernoulli's inequality (Exercise 3.1.24). Thus $f(b) > k$. Since f is continuous, we conclude that there exists $c \in (0, b)$ such that $f(c) = k$. Thus $c^n = k$ and c is an nth root of k.

5.3.8 EXAMPLE The intermediate value theorem is really a very powerful result that can be useful in a variety of settings. To illustrate this diversity, let C be any bounded closed subset of the plane. (For a subset of the plane to be bounded it must be contained in some circle. For it to be closed it must include all the points that are, roughly speaking, on its "edges.") We claim that there is a square S that circumscribes C. That is, $C \subseteq S$ and each side of S intersects C. Although the details of the proof are beyond our reach, the idea of the proof is as follows. Given any $\theta \in [0, 2\pi]$, let r be a ray from the origin having angle θ with the positive x-axis, as in Figure 3. This ray determines a unique circumscribing rectangle whose sides are parallel and perpendicular to r. Let $A(\theta)$ be the length of the sides parallel to r and let $B(\theta)$ be the length of the sides perpendicular to r. Define $f\colon [0, 2\pi] \to \mathbb{R}$ by $f(\theta) = A(\theta) - B(\theta)$.

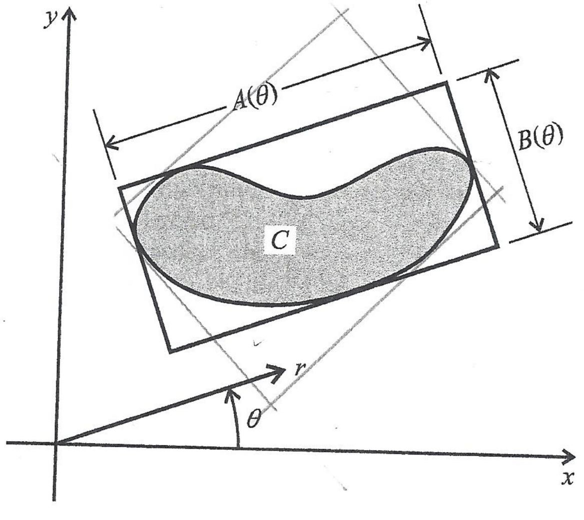


Figure 3 A circumscribing rectangle

If for some particular θ the circumscribing rectangle is not a square, then $A(\theta) \neq B(\theta)$, and let us suppose that we have $f(\theta) = A(\theta) - B(\theta) > 0$. If we replace the angle by $\theta + \pi/2$, then the circumscribing rectangle is unchanged except the labeling of its sides is reversed. That is,

ery positive
et) = x^n.

ice f is con-
) = k. Thus

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et C be any
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> 0. If we
unchanged

$$A\left(\theta+\frac{\pi}{2}\right)=B(\theta) \text{ and } B\left(\theta+\frac{\pi}{2}\right)=A(\theta).$$

Thus $f(\theta+\pi/2)<0$. Now it is reasonable to assume (and not too difficult to verify) that f is a continuous function on $[0, 2\pi]$. (A small change in angle will produce a small change in the lengths of the sides.) Thus by the intermediate value theorem there must be some angle θ_0 between θ and $\theta+\pi/2$ such that $f(\theta_0)=0$. But then $A(\theta_0)=B(\theta_0)$ and the circumscribing rectangle for this angle is a square.

5.3.9 PRACTICE Assuming that $\cos x$ is a continuous function, prove that $x = \cos x$ for some x in $(0, \pi/2)$.

We conclude this section with a theorem that combines two earlier results. Further properties of continuous functions are included in the exercises.

5.3.10 THEOREM Let I be a compact interval and suppose that $f: I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.

Proof: Corollary 5.3.3 implies that there exist x_1 and x_2 in I such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in I$. Let $m_1 = f(x_1)$ and $m_2 = f(x_2)$. Then $f(I) \subseteq [m_1, m_2]$. If $m_1 = m_2$, then $f(I) = \{m_1\} = [m_1, m_2]$, and we are done. If $m_1 < m_2$ and $k \in (m_1, m_2)$, then Theorem 5.3.6 implies that $k = f(c)$ for some c between x_1 and x_2. Thus $(m_1, m_2) \subseteq f(I)$. Finally, since m_1 and m_2 are in $f(I)$, we have $[m_1, m_2] \subseteq f(I)$. Hence $f(I)$ is the compact interval $[m_1, m_2]$. ♦

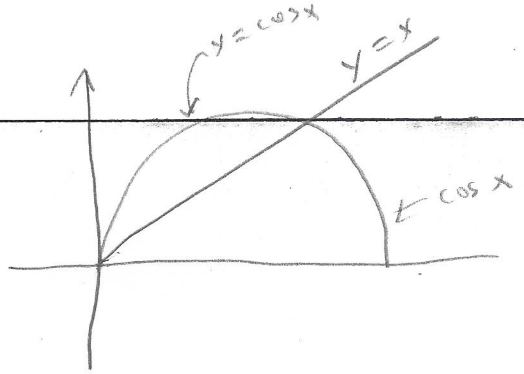
Review of Key Terms in Section 5.3	
Bounded function	Intermediate value theorem

ANSWERS TO PRACTICE PROBLEMS

5.3.1 There are many possibilities. A simple one is $f(x) = 1/x$ for $x \in D = (0, 1)$. Then $f(D) = (1, \infty)$.

5.3.4 $f(x) = x$ is one possibility.

5.3.9 Let $f(x) = x - \cos x$. Then $f(0) = -1$ and $f(\pi/2) = \pi/2$. Since f is continuous, there exists x_0 in $(0, \pi/2)$ such that $f(x_0) = 0$. That is, $x_0 = \cos x_0$.



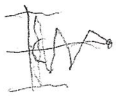
5.3 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.
   - (a) Let D be a compact subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.
   - (b) Suppose that $f: D \to \mathbb{R}$ is continuous. Then, there exists a point x_1 in D such that $f(x_1) \geq f(x)$ for all $x \in D$.
   - (c) Let D be a bounded subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then $f(D)$ is bounded.

2. Mark each statement True or False. Justify each answer.
   - (a) Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(c) = 0$.
   - (b) Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) \leq k \leq f(b)$. Then there exists a point $c \in [a, b]$ such that $f(c) = k$.
   - (c) If $f: D \to \mathbb{R}$ is continuous and bounded on D, then f assumes maximum and minimum values on D.

3. Let $f: D \to \mathbb{R}$ be continuous. For each of the following, prove or give a counterexample. ☆
   - (a) If D is open, then $f(D)$ is open.
   - (b) If D is closed, then $f(D)$ is closed.
   - (c) If D is not open, then $f(D)$ is not open. – $(0, 1/\pi]$



   - (d) If D is not closed, then $f(D)$ is not closed.
   - (e) If D is not compact, then $f(D)$ is not compact.
   - (f) If D is unbounded, then $f(D)$ is unbounded.
   - T (g) If D is finite, then $f(D)$ is finite.
   - (h) If D is infinite, then $f(D)$ is infinite.
   - T (i) If D is an interval, then $f(D)$ is an interval.
   - (j) If D is an interval that is not open, then $f(D)$ is an interval that is not open.

4. Show that $3^x = 5x$ for some $x \in (0, 1)$.

5. Show that the equation $5^x = x^4$ has at least one real solution.

6. Show that any polynomial of odd degree has at least one real root.

7. Suppose that $f: [a, b] \to [a, b]$ is continuous. Prove that f has a **fixed point.** That is, prove that there exists $c \in [a, b]$ such that $f(c) = c$. ☆

8. Suppose that $f: [a, b] \to \mathbb{R}$ and $g: [a, b] \to \mathbb{R}$ are continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$.

21 **9.** Suppose f: $[a, b] \to \mathbb{R}$ is continuous and that $f([a, b]) \subseteq \mathbb{Q}$. Prove that f is constant on $[a, b]$.

10. Suppose that f: $[a, b] \to \mathbb{R}$ is two-to-one. That is, for each $y \in \mathbb{R}$, $f^{-1}(\{y\})$ either is empty or contains exactly two points.

(a) Find an example of such a function.
(b) Prove that no such function can be continuous.

11. (a) Let $p \in \mathbb{R}$ and define f: $\mathbb{R} \to \mathbb{R}$ by $f(x) = |x - p|$. Prove that f is continuous.

(b) Let S be a nonempty compact subset of $\mathbb{R}$ and let $p \in \mathbb{R}$. Prove that S has a "closest" point to p. That is, prove that there exists a point q in S such that $|q - p| = \inf\{|x - p| : x \in S\}$.

23 **12.** Prove Theorem 5.3.2 using the Heine–Borel theorem (3.5.5) and the Bolzano–Weierstrass theorem for sequences (4.4.7) instead of the open-cover property of compactness.

***13.** Let f be a function defined on an interval I. We say that f is **strictly increasing** if $x_1 < x_2$ in I implies that $f(x_1) < f(x_2)$. Similarly, f is **strictly decreasing** if $x_1 < x_2$ in I implies that $f(x_1) > f(x_2)$. Prove the following.

(a) If f is continuous and injective on I, then f is strictly increasing or strictly decreasing.
(b) If f is strictly increasing and if $f(I)$ is an interval, then f is continuous. Furthermore, f^{-1} is a strictly increasing continuous function on $f(I)$.

14. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) Show that f is not continuous at 0.
(b) Show that f has the intermediate value property on $\mathbb{R}$.

15. Let f: $D \to \mathbb{R}$ and let $c \in D$. We say that f is **bounded on a neighborhood** of c if there exists a neighborhood U of c and a number M such that $|f(x)| \leq M$ for all $x \in U \cap D$.

(a) Suppose that f is bounded on a neighborhood of each x in D and that D is compact. Prove that f is bounded on D. ☆
(b) Suppose that f is bounded on a neighborhood of each x in D, but that D is not compact. Show that f is not necessarily bounded on D, even when f is continuous.
(c) Suppose that f: $[a, b] \to \mathbb{R}$ has a limit at each x in $[a, b]$. Prove that f is bounded on $[a, b]$.

22 **16.** A subset S of $\mathbb{R}$ is said to be **disconnected** if there exist disjoint open sets U and V in $\mathbb{R}$ such that $S \subseteq U \cup V$, $S \cap U \neq \varnothing$, and $S \cap V \neq \varnothing$. If S is not disconnected, then it is said to be **connected**. Suppose that S is connected and that f: $\mathbb{R} \to \mathbb{R}$ is continuous. Prove that $f(S)$ is connected. (*Hint*: Use Corollary 5.2.15.)