Kazhdan-Lusztig cells (of Coxeter gps)

$$(W,S) \longrightarrow H = \mathcal{A}\langle C_w : w \in W \rangle$$

$\mathbb{Z}[v,v^{-1}]$

$$\mathcal{A}\langle T_w : w \in W \rangle$$

$\rightarrow C_w = \sum_{y \leq w} P_{y,w} T_y$

KL poly.

Bruhat order

Multiplication rules in the bases:

T: $$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w; \\ (v-v^{-1}) T_w - T_{sw} & \text{if } sw < w; \end{cases}$$ follows from defining relations in H

C: $$C_s C_w = \begin{cases} C_{sw} + \sum_{sy<y<w} \mu_{y,w} C_y & \text{if } sw > w; \\ (v+v^{-1}) C_w & \text{if } sw < w. \end{cases}$$ follows from further facts about H.

where $\mu_{y,w}$ is the coefficient of v^{-1} in $P_{y,w}$ ($\mu_{y,w}$ may be 0).

Kazhdan-Lusztig cells.

Def (KL order) For $x, y \in W$, we

- declare $x \prec_L y$ if C_x appears (w/ nonzero coeff) in $C_s C_y$ for some s; ('right' version: $x \prec_R y$ if C_x appears in $C_y C_s$ for some s.)
- define $x \leq_L y$ if $\exists$ a seq. $x = w_1, w_2, \ldots, w_n = y$ s.t. $w_i \prec w_{i+1}$ $\forall i \in \{1, \cdots, n-1\}$ i.e., if

$$x = w_1 \prec_L w_2 \prec_L \cdots \prec_L w_{n-1} \prec_L w_n = y$$

(so $\leq_L$ is just the transitive closure of $\prec_L$)

· define $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$.

Rmk: Then $\sim_L$ is an equivalence relation on W.

The corresponding equivalence classes are called the <u>left Kazhdan-Lusztig cells</u> of W.

Right (KL) cells can be defined similarly. So can two-sided KL cells. $\left(x \leq_{LR} y \text{ if } x \leq_L y \text{ or } x \leq_R y \xrightarrow[\text{closure}]{\text{transitive}} \leq_{LR} \right.$

$\left. \longrightarrow \sim_{LR} \right)$

Each two-sided cell must be a union of left cells.

Interesting facts / questions

— If y can be obtained from x by a left star operation, then $x \underset{L}{\sim} y$.

— If $x \underset{L}{\sim} y$, then $R(x) = R(y)$.

— <u>Q:</u> When is $FC(w)$ a union of two-sided cells?

<u>A:</u> Known, but highly nontrivial.

Example. Dihedral systems. $\underset{1}{\circ}\overset{m}{\text{———}}\underset{2}{\circ}$

Facts: (1) $C_w = \sum_{y \le w} v^{\ell(y)-\ell(w)} T_y.$

(2)

$C_1 \cdot C_e = C_1$

$C_2 \cdot C_e = C_2$

$C_1 C_{w_0} = (v+v^{-1}) C_{w_0}$

$C_2 C_{w_0} = (v+v^{-1}) C_{w_0}$

$$\left.\begin{aligned}
C_1 \cdot C_2 &= C_{12} \\
C_2 \cdot C_{12} &= C_{212} + C_2 \\
C_1 \cdot C_{212} &= C_{1212} + C_{12} \\
C_2 \cdot C_{1212} &= C_{21212} + C_{212} \\
&\vdots \\
C_2 \cdot C_{\underbrace{\cdots 1212}_{m-1}} &= C_{\underbrace{w_0}_{m}} + C_{\underbrace{\cdots 1212}_{m-2}}
\end{aligned}\right\}$$

(Ex). The left cells are $\{e\}$, $\{w_0\}$ and

$\{2, 12, 212, 1212, \ldots\}$,

$\{1, 21, 121, 2121, \ldots\}$.

Q: (1) Can we detect left/right descent and left/right cell information for FC elts from TL diagrams?

Lusztig–Mathas Involutions.

<u>Thm.</u> Let (W,S) be a finite Coxeter system. Let w_0 be the longest elt in W. Let $w \in W$ and suppose $T_{w_0} C_w = \sum_{y \in W} \alpha_{y,w} C_y$. Then there exists a unique elt y s.t. $\alpha_{y,w} \neq 0$ and $y \sim_L w$.

Moreover, if we denote the unique elt by $\lambda(w)$, then $\lambda(\lambda(w)) = w \ \forall w$.

Q: If $TL(W,S)$ has a diagram realization and $w \in FC(W)$, can we describe $\lambda(w)$ diagrammatically? Key computation: $T_{w_0} C_w$.