Last time: Hecke algebra $H / \mathbb{Z}[v, v^{-1}]$, given by gen. and rels. $\leftarrow (W, S)$

gens : $\{T_s : s \in S\}$

rel : $(T_s - v)(T_s + v^{-1}) = 0 \quad \forall s$

$$T_s T_t T_s \cdots = T_t T_s T_t \cdots$$

Fact: $\{ T_w := T_{s_1} T_{s_2} \cdots T_{s_q} \mid w \in W \}$ is a basis (standard basis)

$w = s_1 \cdots s_q$ reduced

To define a hom from an algebra A given by gens and rels to an algebra B, it suffices to define a function on the gens of A and check that the function respects the relations.

Today. — Kazhdan-Lusztig objects (KL)

— The Temperley-Lieb algebra(s) in type A.

KL objects. Recall that in H, T_s is inv. $\forall w \in W$. with

$$T_s^{-1} = T_s - (v - v^{-1}) \quad \forall s.$$

Claim: $\exists$ an alg hom $\overline{} : H \to H$ sending v to v^{-1}, $(v^{-1}$ to $v)$ and $\overline{T_s}$ to T_s^{-1} $\forall s \in S$.

(unital, i.e., sends 1 to 1)

Pf: It suffices to check that the relations in H are respected. $T_s - (v - v^{-1})$.

[quadratic] $(T_s - v)(T_s + v^{-1}) = 0$. so we need $(\overline{T_s} - \overline{v})(\overline{T_s} + \overline{v^{-1}}) = 0$, i.e.,

$\Longrightarrow$ $(T_s - v + \cancel{v^{-1}} - \cancel{v^{-1}})(T_s - \cancel{v} + v^{-1} + \cancel{v}) = 0$ ✓

[braid] We have $T_s T_t T_s \cdots = T_t T_s T_t \cdots$, so we need

$$\overline{T_s}\,\overline{T_t}\,\overline{T_s} \cdots = \overline{T_t}\,\overline{T_s}\,\overline{T_t} \cdots,$$

i.e., $T_s^{-1} T_t^{-1} T_s^{-1} \cdots T_u^{-1} = T_t^{-1} T_s^{-1} T_t^{-1} \cdots T_v^{-1}$, u, v the proper elt in $\{s,t\}$

(usual fact: $a^{-1} b^{-1} = (ba)^{-1}$ $\forall a, b$ in algebra A)

$$LHS = (\underbrace{T_u \cdots T_s T_t T_s}_{m(s,t)})^{-1} = (T_{w_0})^{-1} = (T_v \cdots T_t T_s T_t)^{-1} = RHS.$$

$w_0 = u \cdots sts = v \cdots tst$, longest

$\Downarrow$ elt in $W_{\{s,t\}}$

$T_{w_0} = T_u \cdots T_s T_t T_s = T_v \cdots T_t T_s T_t$

Done!

② Note that $\overline{T_w} = T_{w^{-1}}^{-1}$.

Note: ① The map $\overline{}$ is an involution in the sense that $(\overline{})^2 = \mathrm{Id}$; it's usually called the bar involution.

Thm. There $\exists$ a unique $\mathcal{A}$-basis for $\mathcal{H}$ of the form $\{C_w : w \in W\}$
$\mathbb{Z}[v, v^{-1}]$

s.t. (a) $\overline{C_w} = C_w$. (bar invariance)

(b) $\forall w \in W$, $C_w = T_w +$ (a linear combination of terms of the form $p_{y,w} \cdot T_y$ where $y < w$ in the Bruhat order on W and $p_{y,w} \in \mathbb{Z}[v^{-1}]$.)

same partial order on W

[Corollary 2.2.3. BB]

('unitriangularity', since the change-of-basis matrix

$$\begin{array}{c} w_1 < w_2 < \cdots < w_k \cdots \\ \left[\quad [C_w]_T \quad \right] \\ w^{\text{th}} \text{ column} \end{array}$$

is upper-Δ w/ 1's on the diag if we order W in a way compatible w/ $<$)

Def. The basis $\{C_w\}$ is called the Kazhdan-Lusztig basis of $\mathcal{H}$.
The elts $p_{y,w} \in \mathcal{A}$ are the Kazhdan-Lusztig polynomials.

generally hard to compute

Baby examples.

(1) $\forall s \in S$. $C_s = T_s + v^{-1}$.

Pf: By uniqueness it suffice to show that $(T_s + v^{-1})$ satisfies (a) and (b).

Bar inv: $\overline{T_s + v^{-1}} = T_s^{-1} + v = T_s - v + v^{-1} + v = T_s + v^{-1}$.

Uni-Δ: obvious. $v^{-1} = v^{-1} T_e$ and $e < s$.

Note: Together w/ 1, $\{C_s : s \in S\}$ generate $\mathcal{H}$ as an $\mathcal{A}$-algebra.

They are known as the KL generators of $\mathcal{H}$.

(2). Ex: For a dihedral system (W, S), $S = \{s, t\}$ w/ $m(s,t) = m$,

$$C_w = \sum_{y \leq w} \left(v^{l(y) - l(w)}\right) T_y.$$

Temperley-Lieb algebras in Type A_n $\longrightarrow$ $TL(A_n)$

We'll see three incarnations of $TL(A_n)$, all isomorphic.

I. As a quotient $TLQ(A_n)$ of $H(A_n)$:

works for all Coxeter systems

The ideal to quotient $H(A_n)$ by is

$$I(A_n) = \left\langle C_{w_0(s,t)} : \begin{array}{l} s,t \in S \\ m(s,t) \geq 3 \end{array} \right\rangle$$

where $w_0(s,t)$ is the longest elt in $\langle s,t \rangle$.

$$\stackrel{A_n}{=} \langle C_{s_i s_{i+1} s_i} : 1 \leq i \leq n \rangle. \quad (\{s,t\} = \{s_i, s_{i+1}\})$$

Def: $TLQ(A_n) = H(A_n) / I(A_n)$

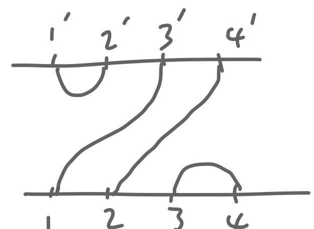
II. As an algebra $TLP(A_n)$ given by gens and relations As follows.

$$TLP_\delta(A_n) = \left\langle E_1, \dots, E_n \;\middle|\; \begin{array}{ll} E_i^2 = \delta E_i & \forall i,\ \delta = v+v^{-1} \\ E_iE_j = E_jE_i & \text{if } |i-j|>1 \\ E_iE_jE_i = E_i & \text{if } |i-j|=1 \end{array} \right\rangle$$

III. As a 'diagram algebra' $TLD_\delta(A_n)$ where a basis (over A) is

$D = \{$ "crossingless pairings of $(n+1)$ points on a north-side plus $(n+1)$ points on a south-side" $\}$

e.g. $n=3$



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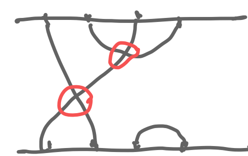
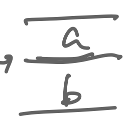


✗

and multiplication is "stacking + straightening + isotopy (extend lin.)

eval. each O" as $\delta = v+v^{-1}$: $ab \to$

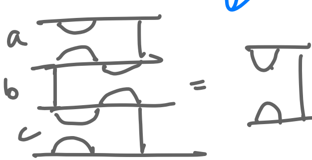


, e.g.

$u_1u_2u_1 = u_1$

abc

//



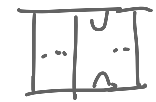
Thm (EX). The three algebra are isomorphic. i.e.

$$TLQ(A_n) \cong TLP_\delta(A_n) \cong TLD_\delta(A_n) \quad \text{when } \delta = v+v^{-1}$$

$TLQ(A_n) = H(A_n)/I(A_n)$,

$TLP_\delta(A_n)$: gen: E_i by def

$TLD_\delta(A_n)$: gen:



$i \quad i{+}1$

$= U_i$

so the KL gen $\{C_i\}$ of $H(A_n)$ descend to gen. $\{[C_i]\}$ of $TL(A_n)$, $(C_i := C_{s_i})$

EX: (1). Show that $U_1, U_2, \dots, U_n$ generate $TLD(A_n)$.

(2). $[C_i] \rightleftarrows E_i \rightleftarrows U_i$ all define algebra homomorphisms. (which are then isomorphism)

Goal for later: Implement the diagram algebra and its variations in other types.