Hecke algebras.

Theme: dealing w/ algebraic objects given by generators and relations.

old example: Coxeter gps.

new examples: gp algebras, Hecke algebras.

Temperley-Lieb algebras. diagram algebras

Def. An algebra (over $\mathbb{C}$ (k, more generally)) is a $\mathbb{C}$ (k)-vector space A with a $\mathbb{C}$ (k)-bilinear associative multiplication.

$$(u v) \cdot w = u \cdot (v \cdot w)$$

$\forall\, u, v, w \in A,\ c, d \in \mathbb{C}$.

$$(cu + dv) \cdot w = c\,(u \cdot w) + d\,(v \cdot w)$$

$$u \cdot (cv + dw) = c\,(u \cdot v) + d\,(u \cdot w)$$

Examples. ① Given any gp G, we may form its gp algebra $\mathbb{C}G$ over $\mathbb{C}$.

As a vector space. $\mathbb{C}G$ is defined to be the span of a basis

$$B = \{ v_g : g \in G \} = \{ [g] : g \in G \} = \{ g : g \in G \}.$$

The multiplication on $\mathbb{C}G$ is defined by setting

$$v_g \cdot v_h = v_{gh}$$

(So $\mathbb{C}G$ is not commutative when G is not abelian)

and extending linearly.

Prop. The above procedure yields a $\mathbb{C}$-algebra.

(0) $\mathbb{C}$ is a $\mathbb{C}$-algebra of dimension 1 (with the usual multiplication); $\mathbb{C}[x_1, x_2, \cdots, x_n]$ is a $\mathbb{C}$-algebra of dim $\underline{\underline{\infty}}$.

Rmk: Both the examples are unital algebras in the sense that there's a multi. identity 1 in the algebra ($1 = v_e$ in $\mathbb{C}G$).

$\underbrace{\text{multi. identity}}_{\text{'unit'}}$

More on (1). $\mathbb{C}G$.

- We may define an algebra by generators and relations

- Prop / Fact: When G is a Coxeter gp $G = \langle s \in S : (st)^{m(s,t)} = 1,\ s^2 = 1 \ \forall s,t \in S \rangle$.

$\mathbb{C}G$ is (isomorphic to) the algebra given by the presentation

$$\langle v_s \in S : (v_s)^2 = 1,\ (v_s v_t)^{m(s,t)} = 1 \ \forall s,t \in S \rangle.$$

Rmk: (a) In addition to gp algebras, we also have gp rings, e.g.

$\underline{\underline{\mathbb{Z}G}}$. elt: $\mathbb{Z}$-linear combinations of formal 'basis elts' v_g, $g \in G$.

multiplication: extended by $\mathbb{Z}$-linearity from the rule

$$v_g \cdot v_h = v_{gh}$$

E.g. S_3. $(2v_{s_1} + 3v_{s_2}) \cdot v_{s_1 s_2 s_1} = 2 \cdot (\underline{v_{s_1}} \cdot \underline{v_{s_1 s_2 s_1}}) + 3 \cdot (\underline{v_{s_2}} \cdot \underbrace{v_{s_1 s_2 s_1}}_{= s_2 s_1 s_2})$

$$= \boxed{2\, v_{s_2 s_1} + 3 \cdot v_{s_1 s_2}}$$

(b). In fact, in addition to algebras over fields, we can define algebras over (commutative) rings. e.g. $\mathbb{Z}G$ is a $\mathbb{Z}$-algebra.

eg. $A = \mathbb{Z}[v, v^{-1}]$ is a $\mathbb{Z}$-algebra.

Hecke algebras (a.k.a. Iwahori-Hecke algebras).

Def. The <u>Hecke algebra</u> of a Coxeter system (W, S, m)

is the unital algebra H over $\mathbb{Z}[v, v^{-1}]$ ('base ring')

Laurent polynomials.

e.g. $3v^2 - v + 5 - v^{-1} + 7v^{-10}$

generated by $\{T_s : s \in S\}$

subject to relations

(1) $(T_s - v)(T_s + v^{-1}) = 0$ $\quad \forall s \in S$ (quadratic rel)

(2) $\underbrace{T_s T_t T_s \cdots}_{m(s,t)} = \underbrace{T_t T_s T_t \cdots}_{m(s,t)}$ $\quad \forall s, t \in S,\ s \neq t$

(braid rel) $\square$

More on (1). $(T_s - v)(T_s + v^{-1}) = 0$

$$T_s^2 - vT_s + v^{-1}T_s - 1 = 0$$

$(T_s - v + v^{-1})T_s = 1$

$$T_s^2 = (v - v^{-1})T_s + 1.$$

'degree reduction'

$$T_s(T_s - v + v^{-1}) = 1$$

so T_s is invertible, w/ $T_s^{-1} = T_s - v + v^{-1}$.

If we specialize v to 1, we get

'H is a deformation of $\mathbb{Z}G$'.

$$(T_s - 1)(T_s + 1) = 0$$

In fact, in this case we recover $\mathbb{Z}G$, i.e.,

i.e. $T_s^2 = 1$, $\quad s \mapsto T_s$ is an iso from $\mathbb{Z}G$ to H.

More on (2). $T_s T_t T_s \cdots = T_t T_s T_t \cdots$

The braid relations (+ Matsumoto's Thm) guarantees that there's no ambiguity if we define T_w for each $w \in W$ to be

$$T_w := T_{s_1} T_{s_2} \cdots T_{s_q}$$

where $s_1 s_2 \cdots s_q$ is any choice of reduced word for w.

Reason for well-definedness: if $s_1' s_2' \cdots s_q'$ is another reduced word for W. then it's related to $s_1 s_2 \cdots s_q$ by braid relations by Matsumoto's Thm, but then (2) guarantees the 'T-products' are also equal in H.

e.g. A_3 $\quad w = \underline{s_1 s_3} s_2 s_1 = s_3 \underline{s_2 s_1 s_2}$

$$s_3 s_1 s_2 s_1$$

$$T_w = T_{s_1} T_{s_3} T_{s_2} T_{s_1} \overset{(2)}{=} T_{s_3} T_{s_1} T_{s_2} T_{s_1}$$

$$T_w = T_{s_3} T_{s_2} T_{s_1} T_{s_2} \overset{(2)}{=} T_{s_3} T_{s_1} T_{s_2} T_{s_1}$$

In particular every elt in H is a unique $\mathbb{Z}[v, v^{-1}]$ lin comb. of $\{T_w : w \in W\}$.

Thm: The set $\{T_w : w \in W\}$ is "$\mathbb{Z}[v, v^{-1}]$-basis" for H.

(H is a free $\mathbb{Z}[v, v^{-1}]$-module over $\{T_w : w \in W\}$)

Rmk: (a) The fact that $\{T_w : w \in W\}$ spans H is easy;

the fact that the set is lin. ind. over $\mathbb{Z}[v,v^{-1}]$ is harder.

e.g. A_3. a random elt in H is

$$2\, T_{s_1} \cdot T_{s_2} \cdot \underbrace{T_{s_1} \cdot T_{s_2} \cdot T_{s_1}}_{*} + T_{s_3} \cdot (3T_{s_2}) \cdot (4 \cdot T_{s_2})$$

Ex. finish this computation

$$= 2\, T_{s_1} \cdot (T_{s_2} \cdot T_{s_1 s_2 s_1}) + 12 \cdot T_{s_3} \cdot (T_{s_2} \cdot T_{s_2}) = \cdots$$

In general (E.x.)
(if w is reduced)

$$\boxed{T_s \cdot T_w = \begin{cases} T_{sw} \quad \text{reduced} & \text{if } sw > w. \quad (*) \\ (v-v^{-1})T_w - T_{sw} & \text{if } sw < w \quad (**) \end{cases}}$$

$$(**) \quad T_s \cdot T_w = T_s \cdot T_{sv} = T_s \cdot T_s \cdot T_v = [(v-v^{-1})T_s - 1]\,T_v = (v-v^{-1})\,T_{sv} - T_v$$

(b). Note that $T_e = 1$, the unit in H.

(empty products are 1 by convention)

(c) The set $\{T_w : w \in W\}$ is called the <u>standard basis</u> for H.

<u>Def.</u> Let A, B be algebras over a commutative ring R $\left(\begin{array}{l} \text{e.g. } R = \mathbb{Z}[v, v^{-1}] \\ R = \mathbb{C} \end{array} \right)$.

Then an algebra homomorphism from A to B is a $\overset{R\text{-linear}}{\text{map}}$ $\varphi : A \to B$ that respects multiplication, i.e., it's a map φ s.t.

$$\varphi(cv_1 + dv_2) = c\,\varphi(v_1) + d\,\varphi(v_2) \quad \forall c, d \in R,\ v_1, v_2 \in A.$$

and $$\varphi(v_1 \cdot v_2) = \varphi(v_1) \cdot \varphi(v_2) \quad \forall v_1, v_2 \in A.$$

A homomorphism is called an <u>isomorphism</u> if it's bijective.

Rmks: (a). These notions are similar to linear transformations / isomorphisms, gp hom / isos you've seen.

(b). When A (the source of the map) is given by generators and relations, say $A = \langle a \in S : \text{rels} \rangle$. we may get a homomorphism from A to B as follows. eg $aba = a$

① Specify how to map each gen $a \in S$ of A to an elt in B. f

② if the assignment from ① 'respects' all the relations in A. $f(a)f(b)f(a) = f(a)$

then it's guaranteed that we can extend the assignment f to a algebra homomorphism $\varphi : A \to B$.

(c) There are analogs of the fact in (b) (how you can induce a map from an object given by a presentation from a 'nice' map on the generating set) for other types of algebraic objects like groups. These facts can be unified in the framework of category theory, via so-called universal properties of free objects and quotient objects.