A bit more on star operations: Star reducibility of elements

Prop 1. Let $w \in W$ be FC. Let $s, t \in S$ with $m(s,t) \geq 3$.

Then we can apply a left lower star operation on w w.r.t. $\{s,t\}$ that removes s iff

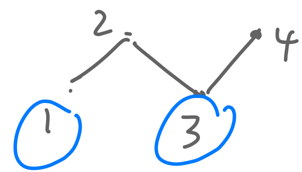
① $s \in \mathcal{L}(w)$ $\boxed{\text{remove}(s,w)}$

② $t \in \mathcal{L}(u)$ where u is obtained by removing from w the first s.

So that the decomposition is not the only st-string $s \cdot^{I} w$ which we can not reduced

Recall: for an FC elt w, $\mathcal{L}(w) = \{$ minimal elts in $H(w)\}$.

e.g. A_5. $w = \underbrace{13}_{\mathcal{L}(w)}24$.



Corollary 2. Let $w \in W$ be FC. Let $s \in S$. Then there is a left lower star operation we can perform on w that removes s iff

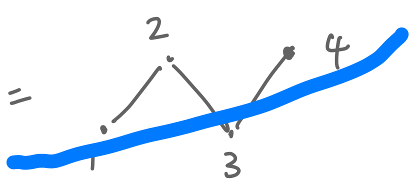
① $s \in \mathcal{L}(w)$

② $t \in \mathcal{L}(u)$ for some $t \in S$ w/ $m(s,t) \geq 3$ where u is the result of removing the first s from w.

Corollary 3. Let $w \in W$ be FC. Let $s \in S$. Suppose $s \in \mathcal{L}(w)$.

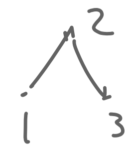
Then there is no lower left star operation that removes the first s from w iff no neighbor t of s in u (the result of removing the s from w) is in $\mathcal{L}(u)$, i.e., if every neighbor t of s in u is blocked by another neighbor of t to the left in w.

e.g.: A_5.

$w =$ 

$w = 1324$

$= 31 \vdots 24$.

$w_2 =$ 

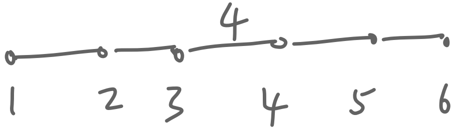
$\{s,t\} = \{3,4\}$. $\longrightarrow$ can get ${}_*w$

$\{s,t\} = \{3,2\}$ $\longrightarrow$ ${}_*w$ is not defined

Corollary 4. Let $w \in W$ be FC. Then we can apply a lower left star operation on w iff for some $s \in \mathcal{L}(w)$, $\mathcal{L}(\text{remove}(s,w))$ contains a neighbor t of s, iff $\exists$ adjacent generators $s, t \in S$ s.t. $s \in \mathcal{L}(w)$, s is the only neighbor of t appearing to the left of the first t in w.

All the above can be visualized in heaps.

Chase: "You can remove an s via a star reduction exactly when s is the sole elt covered by something!"

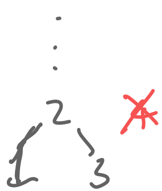
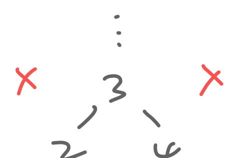


Suppose w is FC and not star reducible from the left. Say $w = \underbrace{w_1}_{\mathcal{L}(w)} w_2 \cdots w_k$ in the Cartier – Foata form.

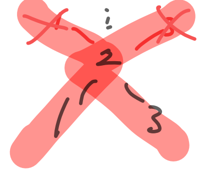
Assume w is not a 'commuting product'. Then w_2 is not empty. (want: w is reducible from the right.)

Then. every generator t in w_2 must have at least two neighbors in w_1.

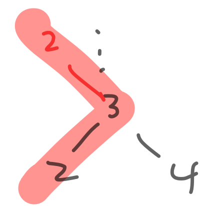
Rephrasing of Thomas' Case 1: $|w_1| = 2$.

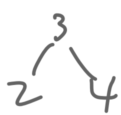
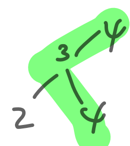
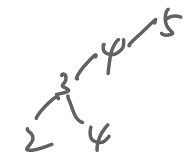
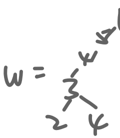
$w =$  or $w =$  or left-right mirror images of these cases.

all things in the first two layers are drawn.

If w starts w/ $w =$  in the first two layers,

then w must be exactly 132

If w starts w/ $w =$  in the first two layers,

then $w =$  or $w =$  or $w =$  or $w =$ .

The reduction from Case 2 to Case 1 is the same as before.

Endgame: in all these cases, w is star-reducible from the right.