Star operations.

· **Parabolic subgps.** Let (W, S) be a Coxeter system. Each subset $I \subseteq S$ generates a subgp W_I consisting of all elts whose reduced words only involve generators from I.

Such a gp is called a parabolic subgp generated by I.

e.g. $S_5 = A_4$ $\quad S = \{s_1, s_2, s_3, s_4\}$

$$I = \{s_1, s_2\} \rightsquigarrow W_I = \langle s_1, s_2 \rangle = \left\{\begin{matrix} e, s_1, s_2, \\ s_1 s_2, s_2 s_1, \\ s_1 s_2 s_1 \end{matrix}\right\} = A_2$$

$= \text{perm. that fix } 4 \text{ and } 5.$

$$J = \{s_1, s_3\} \longrightarrow W_J = \langle s_1, s_3 \rangle = \left\{\begin{matrix} e, s_1, s_3 \\ s_1 s_3 = s_3 s_1 \end{matrix}\right\}$$

$\overset{?}{=}$ perm. fixing 5 and keep $(2,3)$, $(4,5)$ in order

Rmk:

- Fact: (W_I, I) is also a Coxeter system.
- Restriction from W to a parabolic subgp behaves well with all Coxeter-group statistics we'd encounter, e.g.

  if $w \in W_I \subset W$, then $\ell_{W_I}(w) = \ell_W(w)$.

  e.g. $a_{W_I}(w) = a_W(w)$ for our a-function (later).

- Coset decomposition.

Prop. Let (W, S) be a Coxeter system. Let $J \subset S$. Then every $w \in W$ can be written in a unique way as

$$w = w^J \cdot w_J$$

where $w_J \in W_J$ and w^J has no right descent from J. In this case, $\ell(w) = \ell(w^J) + \ell(w_J)$. $\square$

e.g. A_5. $w = s_1 s_3 s_2 s_1 s_4 s_5$, $J = \{ s_2, s_5 \}$.

In fact, we can always get the coset decomposition

$$w = w^J \cdot w_J$$ via a greedy algorithm

start with $w' = w$, use "if $w's < w'$, $s \in R(w')$"

to 'pull' s out from w' if $s \in J$ whenever possible.

$w' = w'' s$.

$w' \leftarrow w''$

$w = 132145$.

$5 \in R(w')$? Yes.

$w' = 13214 \cdot 5 \rightarrow (13214) \cdot 5$

new w'

$5 \in R(w')$? No.

13214. $2 \in R(w')$? Yes. $w' = 13214 = 31214 = 32124 = 32142 = (3214)\,2$

new w'

3214. $5 \in R(w')$? No

$2 \in R(w')$? No. $\Big\} \longrightarrow$ stop coz. $R(w') \cap J = \emptyset$. $\Rightarrow w_J = 25$, $w^J = 3214$.

You can read Section 2.4 of [BB] for more details like uniqueness of the factorization.

Star operations

Now we restrict to parabolic subgps W_J of (W,S) where J contains two non-commuting generators, i.e., $J = \{s,t\}$ $m(s,t) \geq 3$.

In fact, we'll also restrict to only f.c. elts. so we'll consider coset decomp of fc elts w/ two noncommuting generators.

Note that if $w \in W$ and $J = \{s, t\}$ with $\overset{m}{\overset{\shortparallel}{m(s,t)}} \geq 3$, then

w must be of one of the following mutually exclusive forms:

(1) $w = w^J \cdot e$, i.e. $w = w^J \cdot w_J$ where $w_J = e$

(2) $w = w^J \cdot w_J = w^J \cdot \underbrace{stst\ldots}_{} = w^J \cdot \underbrace{tsts\ldots}_{}$ (impossible if w is FC.)

m letters in each

(3). w is one of $\underset{x_1}{\underset{\shortparallel}{w^J \cdot s}}$, $\underset{x_2}{\underset{\shortparallel}{w^J \cdot st}}$, $\underset{x_3}{\underset{\shortparallel}{w^J \cdot sts}}$, ---, $\underset{x_{m-1}}{\underset{\shortparallel}{w^J \cdot \underbrace{sts\ldots}_{m-1 \text{ letters}}}}$.

(4) w is one of $\underset{y_1}{\underset{\shortparallel}{w^J \cdot t}}$, $\underset{y_2}{\underset{\shortparallel}{w^J \cdot ts}}$, $\underset{y_3}{\underset{\shortparallel}{w^J \cdot tst}}$, ---, $\underset{y_{m-1}}{\underset{\shortparallel}{w^J \cdot \underbrace{tst\ldots}_{m-1 \text{ letters}}}}$.

The elts in (3) or (4) are called right J-strings.

Def: Let $J = \{s, t\}$ where $m(s,t) \geq 3$. Then a <u>right upper star operation w.r.t. J</u> is an operation of the form

$$x_i \longmapsto x_{i+1} \quad \text{or} \quad y_i \longmapsto y_{i+1} \text{ for some } 1 \leq i \leq m-2;$$

a <u>right lower star operation w.r.t. J</u> is an operation of the form

$$x_i \longmapsto x_{i-1} \quad \text{or} \quad y_i \longmapsto y_{i-1} \text{ for some } 2 \leq i \leq m-1.$$

(Thus, star operations w.r.t. J are only partially defined on W: you can only apply it to J-strings, and even then they only make sense if the 'natural addition/deletion' still results in a J-string.)

Rmks: · Given w and J, there are potentially four types of star ops. for which we'll use the following notation:

left lower	$w \longmapsto$	${}_*w$
left upper	$w \longmapsto$	${}^*w$
right lower	$w \longmapsto$	w_*
right upper	$w \longmapsto$	w^*

e.g. A_n. $w = s_3 s_1 s_2 s_1 \implies w^*, w_*$ are not defined

$w = s_3 s_1 s_2 \implies w^*$ is not defined, $w_* = s_3 s_1$

· Prop: If w is FC, then the result of any star operation on w is also FC.