Last time: Coxeter gps. reduced words. reduced word graph

FC elts.

Key Reference: Stembridge 1996. On the FC elts of Coxeter gps.

Def (FC): An elt w in a Coxeter gp is FC if given any two red. words (w_1, w_2) of w, we can (choose to) relate them via only commutation relations. requires checking all reduced words.

Thm (FC criterion) [Stem 96]: w is FC $\iff$ no reduced word of w contains a long braid.

a practical way to check FC.

(we have to use comm relations for $w_1 \to w_2$).

Background (Matsumoto's Thm): w_1 and w_2 are related by braid relations only.

e.g. S_5, $w = \underbrace{s_4 s_2 s_1 s_3 s_4}_{\text{no long braid}} = s_2 s_4 s_1 s_3 s_4 = s_2 s_1 \underline{s_4 s_3 s_4}$

no long braid $\not\Downarrow$ FC

long braid, $\Downarrow$ not FC

Ex. What are the FC elts in S_4? (Natalie)

Today: More FC criteria / tests ; Heaps ; Coxeter diagrams.

Coxeter diagrams / matrices

. Recall that a Coxeter system is a triple

(W, S, m) where $W = \dfrac{\langle S \rangle}{(st)^{m(s,t)} = 1} \quad \forall\, s, t \in S$

eg. $S_4 = \dfrac{\langle s_1, s_2, s_3 \rangle}{\begin{matrix} s_i^2 = 1 \ \forall i \\ s_1 s_3 = s_3 s_1 \\ s_1 s_2 s_1 = s_2 s_1 s_2,\ s_2 s_3 s_2 = s_3 s_2 s_3 \end{matrix}}$

$= \dfrac{\langle S \rangle}{\begin{matrix} s^2 = 1 \quad \forall s \in S \\ \underbrace{sts\ldots}_{m(s,t)} = \underbrace{tst\ldots}_{m(s,t)} \end{matrix}}$

We may therefore encode the system via

① the Coxeter matrix: $S \times S$

$$\begin{array}{c|c} & t \\ \hline s & \begin{bmatrix} & \boxed{m(s,t)} \\ & \\ & \end{bmatrix} \end{array}$$

e.g.

$$\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & & & \\ s_2 & & \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} & \\ s_3 & & & \end{array}$$

$$S_5 \longrightarrow M = \begin{array}{c|c} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix} & \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} \\ \begin{matrix} s_1 & s_2 & s_3 & s_4 \end{matrix} & \end{array}$$

② the Coxeter diagram: $G = (S, E, wt)$

S: vertices, E: $\{s,t\}$ is an edge $\Updownarrow$ $m(s,t) \geq 3$

$\Downarrow$

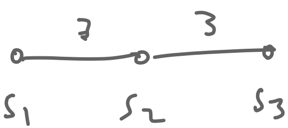
equiv, s is not connected to t $(s \neq t)$ iff $m(s,t) = 2$, iff $st = ts$, iff they commute.

wt: weight labelling of edges. $wt(\{s,t\}) = m(s,t)$

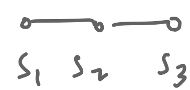
Convention: leave wt-3 edges unlabelled.

Notation: We often call S_{n+1} the Coxeter gp of type A_n.

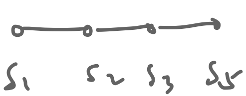
e.g. $S_4 = A_3$ (3 = # gens)



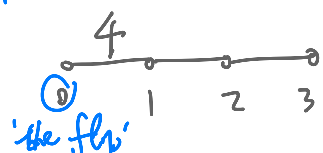
or



S_5



B_4. (4 = # gens; 4 cards.)



More generally, there are frequently used families of Coxeter systems named A_n, B_n, D_n, E_n, F_n, H_n, $I_2(m)$. e.g. $I_2(m)$ $\overset{s}{\circ}\overset{m}{\text{———}}\overset{t}{\circ}$

Ex. ① $|I_2(m)| \leq 2m$. ② Harder: $I_2(m) \cong D_m$

the dihedral gp w/ 2m elts

Heaps. (We'll use A_4 $\overset{a}{\circ}\text{—}\overset{b}{\circ}\text{—}\overset{c}{\circ}\text{—}\overset{d}{\circ}$ as an example.)

Def. (Heap of a word). Given a Coxeter system (W, S, m) and a word $\underline{w} = s_1 s_2 \cdots s_q \in \langle S \rangle$ (where $s_i \in S$ $\forall i$), the heap of $\underline{w}$ is the "labelled poset" whose elts are the indices $1, 2, \ldots, q$, whose covering relation $\prec$ is given by

poset = partially ordered set

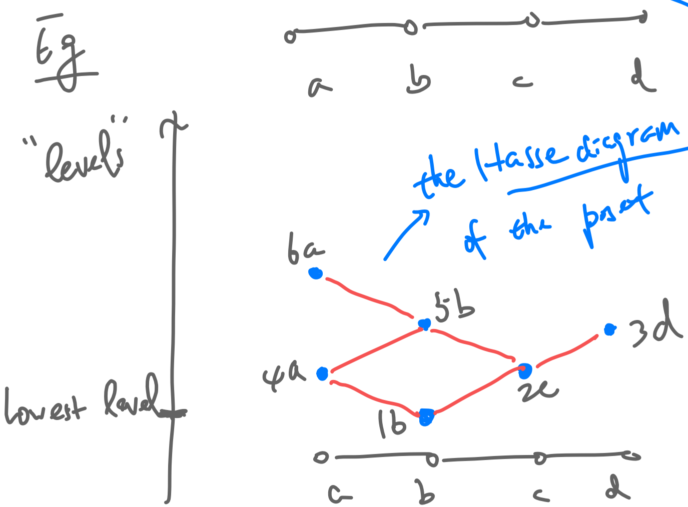
'$s_i <_r s_j$' if $\underline{i<j}$ and $m(s_i, s_j) \geq 3$,

and where the label of an elt i is just s_i.

The heap can be easily visualized: let each letter drop as low as possible from left to right, subject only to the condition that each letter drops higher than each adjacent letter which has dropped.

We might change the reading order at some point.

Eg



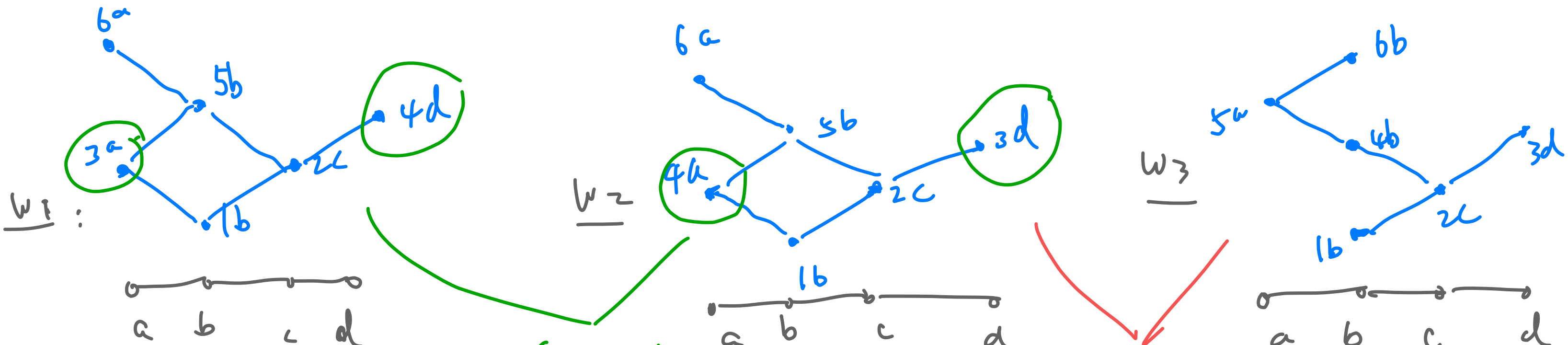
$\underline{w} = b\,c\,d\,a\,b\,a$

$= s_1 s_2 s_3 s_4 s_5 s_6$

E.g.

$$w = bcadba = bcdaba = bcdbab$$

w_1 w_2 w_3.



Same picture!

More precisely, the heaps are isomorphic labelled posets.

Reason: the words differ by only a commutation relation.

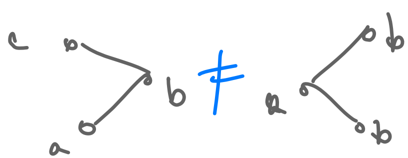
different.

More precisely, the heaps are not isomorphic as labelled posets.

Reason: the words are related by a longer braid rel.

Point: If $w \in W$ is an FC elt, all red words of it are related by only commutation relations, so they all produce isomorphic (the same) heaps, so <u>the heap of an FC elt</u> is well-defined: just take the heap of any red. word of it.

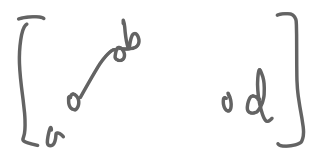
e.g. $aba = bab$



as labelled posets

because aba is not FC.

$abd = adb = dab$

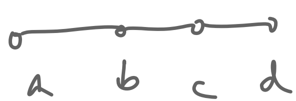


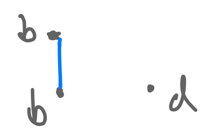
for all three words once we ignore the indices.

Prop. (Heap criterion for FC). Let (W,S) be a Coxeter system. Let $\underline{w} \in \langle S \rangle$. Then $\underline{w}$ is the reduced word of an FC elt in W iff the following conditions hold:

(1) There is no covering relation of the form ss for any $s \in S$.
i.e., in the heap diagram, no 'column' has two connected dots.

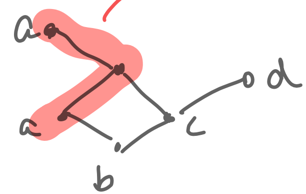
(2) For any $s, t \in S$ with $m(s,t) \geq 3$, there is no convex chain of the form $s - t - s - \cdots$ with $m(s,t)$ elts.

e.g. A_4. 

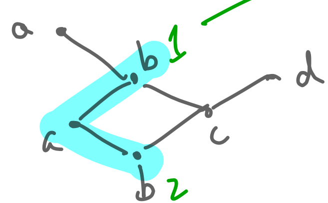
(1) bdb 

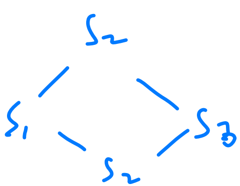
(7). bcadba

convex chain aba $\Longrightarrow$ not FC.



not convex since b1, b2 has the c in between them that's not in the chain.





$s_2 s_1 s_3 s_2$

Rmk: The prop is largely a translation of Stembridge's word criterion for FC, but it has the advantage of just needing to look at the given word itself and not all equivalent words.

Convex chain: a sequence $a_1 - a_2 - \cdots - a_n$ where

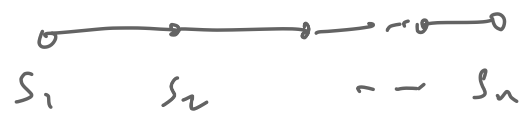
each $a_i - a_{i+1}$ is a covering and for all i, j,

say $i < j$, if b satisfies

$$a_i \le b \le a_j ,$$

then $b \in \{a_i, a_{i+1}, \ldots, a_j\}$.

A new FC criterion (simplified) for A_n.



Claim: $w \in \langle S \rangle$ is the reduced word of an FC elt

$\Updownarrow$

(1) no 'ss'-covering exists in the heap

(2a). No $\begin{matrix} s_1 & \\ & s_2 \\ s_1 & \end{matrix}$ chain, no $\begin{matrix} s_n & \\ & s_{n-1} \\ s_n & \end{matrix}$ chain exists in the heap.

(2b) for each $1 < i < n$, between every two consecutive s_i's in w, there is exactly one s_{i-1} and one s_{i+1}. ($24 | 352$)

To do: ① Use the claim to count FC elts in S_4.

* ② Write a function in sage that takes in a reduced word on $\langle s_1 \ldots s_n \rangle$ and determines if it's an FC elt.

* ②' . . . . takes in a word . . . . . . if it's reduced and FC.

③ Generalize ~~the~~ claim to type B_n