Last time: symm. gps S_n. permutation (functions), diagrams, Coxeter presentation.

$$S_n \cong \left\langle s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \\ s_i s_j = s_j s_i & \text{if } |j-i|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{if } |j-i| = 1. \end{array} \right\rangle$$

What's really going on: RHS defines a 'word gp' whose elts are equivalence classes under the relations. eg. $s_1 s_3 = s_3 s_1$.

This gp turns out to be 'the same' as S_n under the identification $s_i \longleftrightarrow (i, i+1)$. eg. $(12)(34) = (34)(12)$

nontrivial

Intuitively: we identify the two sides with $s_i \longleftrightarrow (i, i+1)$

E.g. (1) $S_n \cong G_n := \left\langle s_1, s_2 \;\middle|\; \begin{array}{c} s_1^2 = s_2^2 = 1 \\ s_1 s_2 s_1 = s_2 s_1 s_2 \end{array} \right\rangle$. Prove $|G_n| \le 6$.

Since the gen s_1, s_2 are their own inverses ($s_1^2 = s_2^2 = 1$), all words in $\langle s_1, s_2\rangle$ are words on s_1, s_2 only (as opposed to needing s_1^{-1}, s_2^{-1}). Since $s_1^2 = s_2^2 = 1 = (\ )$, every word is equivalent to a word without two consecutive repeated letters. In other words, the latter word must alternate in s_1, s_2. Moreover, any alternating string $\underline{s_i s_j s_i} s_j$ where $i \ne j$ is equivalent to $s_j s_i \underline{s_j s_j} = s_j s_i$, whichever is shorter. So all words in G_n are equivalent to one of $(\ )$, s_1, s_2, $s_1 s_2$, $s_2 s_1$, $s_1 s_2 s_1 = s_2 s_1 s_2$, so $|G_3| \le 6$. □

Rmks: (1) Two different 'redundancy'.

$s_1 s_2 s_1 s_2 = s_2 s_1$

reduction (length)

$s_1 s_2 s_1 = s_2 s_1 s_2$

both reduced already

(2) We only proved $|G_n| \leq 6 = |S_3|$. and not really $G_n \cong S_n$

because the latter result requires more gp theory. In particular,

we can't even be sure $|G_n| \geq 6$ at the moment since there may be

tricks we don't know to show, say $s_1 s_2 s_1 = s_1$.

Ex. Prove that $G_4 = \cancel{\langle s_1, s_2, s_3 \rangle \ldots}$ has at most 24 elts.

A slight generalization of S_n.

Consider $B_n = \langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$

equiv class of words without an s_0 behave as in S_n.

$s_i^2 = 1$ $\forall i$

$s_i s_j = s_j s_i$ if $|j-i|>1$ → commutation relation

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ when $i \geq 1$

$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$

} two kinds of braid relations.

Ex: (1) $\|B_2\| \leq 8$; (2) $\|B_3\| \leq 8 \cdot 6$; (3) $\|B_n\| \leq 2^n \cdot n!$

In fact, similar to the comb. realization of S_n as perm. in S_n, → rearranging n-cards

B_n has a combinatorial realization. B_n is the set of ways you can rearrange n cards while allowing flipping.

$s_i \leftrightarrow (i, i+1)$ $i \geq 1$

$s_0 \leftrightarrow$ flipping card 1.

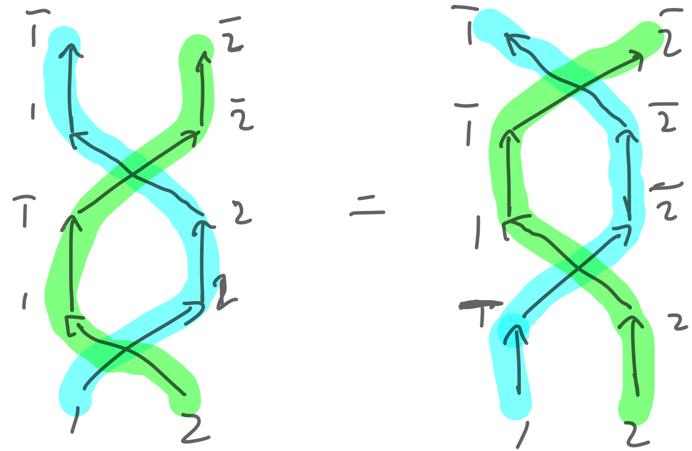
walled wiring diagrams.?

signed

$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0.$

— : the spot is occupied by a card flipped over.

Rule: bars pass along wires



2. Coxeter groups.

a triple (W, S, m)

Def. A Coxeter system is a pair (W, S) where is a generating set for W and W is a gp w/ presentation.

$$W = \left\langle S \;\middle|\; \begin{array}{l} s^2 = 1 \\ \underbrace{sts\ldots = tst\ldots}_{\text{braid relation}} \text{ where both sides have } m(s,t) = m(t,s) \text{ factors.} \end{array} \quad \forall s,t \in S,\ s \neq t \right\rangle$$

Here, m is a function $m: S \times S \longrightarrow \mathbb{Z}_{\geq 1} \cup \{\infty\}$ s.t. $m(s,t) = m(t,s)$; if $m(s,t) = m(t,s) = \infty$, we omit the relation $sts\ldots = tst\ldots$

Notes: · Since $s^2 = t^2 = 1$. if $m = m(s,t) < \infty$, then

$$(st)^m = 1 \iff sts\ldots = tst\ldots$$

e.g. $m=4$. $(st)^4 = 1 \iff stststst = 1 \overset{\cdot t\cdot s\cdot t\cdot s}{} \iff stst \overset{\cdot s\cdot t\cdot s\cdot t}{} = tsts \overset{\cdot s\cdot t\cdot s\cdot t}{}$

So sometimes people write $(st)^{m(s,t)} = 1$ for the braid relations.

In this convention, we can write $W = \langle S \rangle \big/ (st)^{m(s,t)} = 1 \ \forall s,t \in S$

where $m(s,t) = m(t,s) \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $m(s,s) = 1$ and

$m(s,t) \geq 2 \ \forall s \neq t$.

Def. (Elements in W are represented by words, not necessarily uniquely).

Of all words on $\langle S \rangle$ that represent an elt $w \in W$, the ones of minimal length are called the <u>reduced words</u> of w.

Eg. In S_3, $s_1 s_2 s_1 s_2 = s_2 s_1$ represent the same elt,

$s_1 s_2 s_1 s_2$ is not reduced, $s_2 s_1$ turns out to be reduced.

It also turns out that $s_1 s_2 s_1 = s_2 s_1 s_2$ are both reduced words of the same elt.

Theorems

Thm 1. (Recall that $W = \langle S \rangle \Big/ \begin{array}{l} s^2 = 1 \\ (st)^{m(s,t)} = 1 \quad \forall s,t,\ s \neq t. \end{array}$)

(the m's are orders)

(Recall that the order of a gp elt w is the smallest positive integer k s.t. $w^k = 1$. e.g. $\{0, 1, 2, 3, 4, 5\}$ forms a gp under $+$ modulo 6. e.g. $4+5 = 9 = 3$. Then $\underline{4+4+4+4+4+4 = 24 = 0}$ (id) and $4+4+4 = 12 = 0$ but $4 \neq 0$, $4+4 = 8 = 2 \neq 0$. So the order of 4 is 3.)

$4^{\boxed{6}} = \text{id} \not\Leftrightarrow \text{ord}(4) = 6.$

In $W = \langle S \rangle \Big/ \begin{array}{l} s^2 = 1 \\ (st)^{m(s,t)} = 1 \end{array}$, the order of st for $s \neq t$ is exactly $m(s,t)$.

Thm 2. (Matsumoto's Thm) Let $w \in W$. Then every pair of red. words of w can be obtained from each other by a finite sequence of braid relations $sts\ldots = tst\ldots$. (No need to use $s^2 = 1$.)

e.g. S_6:

$$\begin{tikzcd}
s_2 \underline{s_5 s_1} s_4 s_5 s_3 \arrow[d] \arrow[r, equal] & s_2 s_1 s_4 s_5 s_4 s_3 \arrow[d, equal] \\
s_2 \underline{s_1 s_5} \, \underline{s_4 s_5 s_3} \arrow[r] & s_2 s_1 \underline{s_4 s_5 s_4} s_3 .
\end{tikzcd}$$

Def: The <u>reduced word graph</u> of an elt $v \in W$ is the graph whose vertices are the reduced words of w and where two reduced words are connected by an edge if they differ by a single braid move.

(We can 'generate'/recover the whole graph from one vertex.)

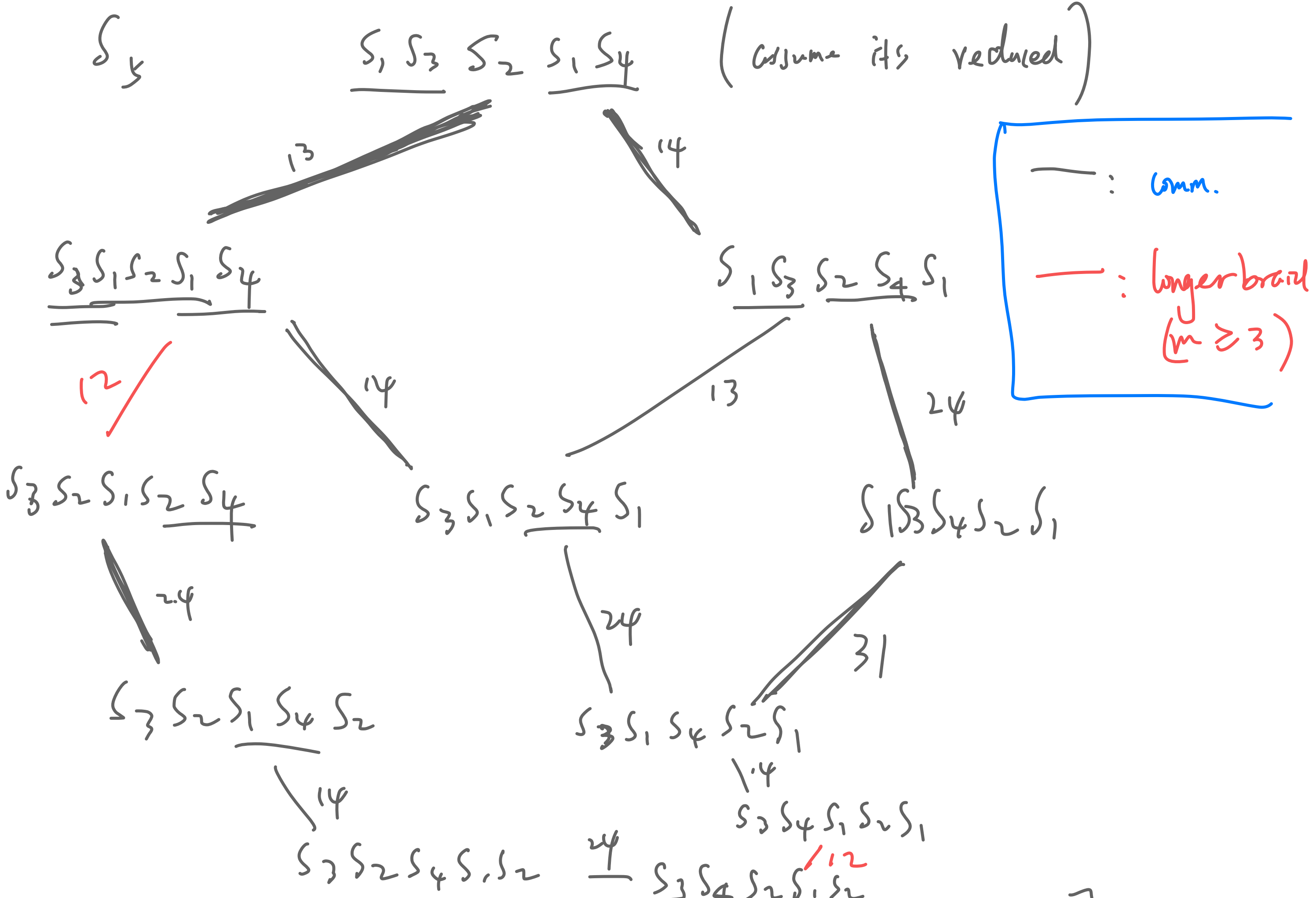
<u>Matsumoto's Thm. rephrased:</u> The reduced word graph of w is always connected.

e.g. S_3, the only reduced words for $s_1 s_2 s_1$ are $s_1 s_2 s_1$ and $s_2 s_1 s_2$ since they are the only words of length 3. So we get

$s_1 s_2 s_1$ ——— $s_2 s_1 s_2$

for the graph. All other elts' graphs have a single vertex.

e.g. S_5



$s_1 s_3 s_2 s_1 s_4$
(assume it's reduced)
13
14
$s_3 s_1 s_2 s_1 s_4$
$s_1 s_3 s_2 s_4 s_1$
12
14
13
24
$s_3 s_2 s_1 s_2 s_4$
$s_3 s_1 s_2 s_4 s_1$
$s_1 s_3 s_4 s_2 s_1$
24
24
31
$s_3 s_2 s_1 s_4 s_2$
$s_3 s_1 s_4 s_2 s_1$
14
14
$s_3 s_4 s_1 s_2 s_1$
12
$s_3 s_2 s_4 s_1 s_2$
24
$s_3 s_4 s_2 s_1 s_2$
: comm.
: longer braid. ($m \geq 3$)

□

Def: An elt $w \in W$ is called fully commutative (FC)

If all its reduced words can be related via only commutation relations (without needing longer braids)

(In the red word graph, this condition says 'any vertices has a connecting path w/ only commutation/gray edges.)

(John Stembridge. 1997. On the FC elts of Coxeter grps.)

Thm. w is FC $\Longleftrightarrow$ $RWG(w)$ has no red edge.

Ex: Find all FC elts in S_3 and S_4.