AA2. Lecture 9.

k: a field

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Last time:

- free vector spaces on a set
- group algebras kG of groups G
- quivers Q and their path algebras kQ

one more def: For any $a \in Q_0$ for a quiver $Q = (Q_0, Q_1)$, the stationary path e_a have length 0; each arrow $\alpha \in Q_1$ has length 1; each path $p = \alpha_k \cdots \alpha_2 \alpha_1$ has length $\sum_{i=1}^{k} \mathrm{len}(\alpha_i)$.

Today:

- examples of alg. homs/isos
- more on ideals: principal ideals

3. Examples of algebra isomorphisms.

(1) $Q = \begin{tikzcd} v \arrow[loop above, "\alpha"] \end{tikzcd}$ $\Rightarrow$ $kQ \cong k[t]$ via the unique lin. map s.t. $t^i \mapsto \alpha^i$ $\forall i \geq 1$ and $1 \mapsto e_v$.

↑ more formally.

kQ: basis: $P = \{e_v, \alpha, \alpha^2, \alpha^3, \dots, \alpha^k, \dots\}$

mult (among basis elts): $\alpha^i \cdot \alpha^j = \alpha^{i+j}$,

$\Big\}$ "just like polynomials"

Ex: Prove the claim. (it reduces to how mult. is respected on basis elts)

Q: $Q = \begin{tikzcd} v \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$ $\overset{?}{\Rightarrow}$ $kQ \cong k[x,y]$

No. $k[x,y]$ is commutative by def ($xy \overset{\text{e.g.}}{=} yx$)

but kQ is not $\alpha\beta \neq \beta\alpha$ by def.

Fact: $kQ \cong k\langle x,y \rangle$, the so-called "free algebra on $\{x,y\}$"....

(2). kQ for $Q = \underset{1}{\cdot} \xleftarrow{\alpha} \underset{2}{\cdot}$ vs. $T_2(k) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in k \right\} \subseteq M_2(k)$

basis: $P = \{e_1, e_2, \alpha\}$

mult:

	e_1	α	e_2
e_1	e_1	α	0
α	0	0	α
e_2	0	0	e_2

Convention: col label as left factor $e_1 \cdot \alpha$

basis: $\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

same

mult:

	E_{11}	E_{12}	E_{22}
E_{11}	E_{11}	E_{12}	0
E_{12}	0	0	E_{12}
E_{22}	0	0	E_{22}

Useful formula: for mult of matrix units, we have $E_{ij} E_{kl} = \delta_{jk} E_{il}$

Kronecker delta, 1 if $j = k$, 0 otherwise

Ex: The unique linear map $\varphi : T_2(k) \to kQ$ with $E_{11} \mapsto e_1$, $E_{12} \mapsto \alpha$, $E_{22} \mapsto e_2$ is an alg. iso.

More generally: kQ for $Q = 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots (n-1) \leftarrow n$ is isomorphic to $T_n(k) = \left\{ \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix} \right\} \subseteq M_n(k)$ as algebras via the map $E_{ij} \mapsto$ the unique path from j to i in Q $(i \leq j)$.

upper Δ

Ex: prove this!

13) $G = \langle g : g^k = e \rangle$

basis: $G = \{e, g, g^2, \ldots, g^{k-1}\}$

mult: $g^i \cdot g^j = g^{i+j} = g^{\overline{i+j}}$

the remainder for $i+j$ when divided by k.

Ex: $$kG \cong k[t] / \langle t^k - 1 \rangle$$

more on ideals in $k[t]$ soon.

pf strategy: Consider the evaluation hom (HW) $\mathrm{Eval}_{t=g} : k[t] \to kG$ with $t \mapsto g$.

Show the hom is surj and has kernel $t^k - 1$, then apply

(4) $\mathbb{R}[x]/\langle x^2+1\rangle \cong \mathbb{C}$

Consider the evaluation hom $\mathrm{Eval}_i : \mathbb{R}[x] \to \mathbb{C}$ w/ $x \mapsto i$.

(e.g. $\underline{a+bx \mapsto a+bi}$, $x^2 \mapsto i^2 = -1$).

$\underbrace{\mathrm{Eval}_i \text{ is surj. } \underline{\text{Ex.}}}$ $\ker(\mathrm{Eval}_i) = \langle x^2+1 \rangle$.

$\Downarrow$ Iso thm.

<u>Consequence:</u> $\mathbb{R}[x]/\langle x^2+1\rangle \cong \mathbb{C}$ as $\mathbb{R}$-algebras.

$\downarrow$

an elegant construction of the complex field !

2. Principal ideals

Lemma: Let A be a k-algebra and let $z \in A$. Then the set

$$Az = \{ az : a \in A \}$$

is a left ideal of A. Similarly, $zA = \{ za : a \in A \}$ is a right ideal of A.

Ex: Prove the lemma.

Def: The ideal Az is called the <u>principal left ideal</u> generated by z.

- - - zA - - - - - - right ideal - - - .

If A is commutative, $zA = Az$, and we denote it by $\langle z \rangle$.

We'll now discuss some principal ideals.

(a) $k[x]$.

Recall that $k[x]$ has a division / Euclidean algorithm: long division,

$\forall f, g \in k[x]$, $\exists!\ q, r \in k[x]$ s.t. $\deg(r) < \deg(g)$ and

$$f = q \cdot g + r$$

A consequence of the algorithm is that every $I \subseteq k[x]$ is a principal ideal, namely, $I = \langle p \rangle$ where p can be taken as any poly in I of minimal deg.

Q: · If $A = kG$, what's Ag for each $g \in G$?
gp

· If $A = kQ$, what's Ae_i for each $i \in Q_0$?
quiver

Next time:
· pf of the consequence
· more on ideals
· Ch 2: modules