AA2. Lecture 8.

01. 28. 2022.

Last time:
- Construction of quotient algebras, well-definedness
- the four isomorphism theorems for algebras

Today:
- more examples of algebras:
  group algebras, path algebras of quivers
- some examples of algebras homs/isos.

1. Group algebras

k : a field

Note: Given any set X, we may construct a k-vector space with X as a (formal) basis : elts of the space are (formal) finite lin. comb of elts of X

(**E.g.** X = {burger, pizza, hotdog}. Then the v.s. contains 2·burgers + 3. pizzas)

The above space is called the free vector space on X and denoted by kX.

Def: Let G be a gp. The <u>group algebra</u> of G is the free vector space kG on G equipped with the multiplication given by

$$\left(\sum_{g\in G}\underline{\alpha_g}\cdot g\right)\cdot\left(\sum_{h\in G}\underline{\beta_h}\cdot h\right)\overset{*}{=}\sum_{g,h\in G}(\alpha_g\beta_h)\cdot\underset{\substack{\cap\\ G}}{(gh)}\quad\forall\,\alpha_g,\beta_h\in k.$$

only finitely many nonzero

Prop1: The above multiplication does make kG a k-algebra.

Example: Take $G = \langle g : g^3 = e \rangle = \{e, g, g^2\}$. Then kG has a basis $\{e, g, g^2\}$ and hence dimension 3. (In general $\dim_k kG = |G|$.)

Here's a multiplication of two typical elts:

$$(\alpha \cdot e + \beta \cdot g) \cdot (\gamma g + \delta g^2) = (\alpha\gamma) \cdot (eg) + (\alpha\delta) \cdot (eg^2) + (\beta\gamma) \cdot (gg) + (\beta\delta) \cdot (gg^2)$$

$$= (\beta\delta) e + (\alpha\gamma) g + (\alpha\delta + \beta\gamma) g^2$$

Pf of Prop 1: We need to show that (*) defines an associative, bilinear, and unital multiplication on the v.s. kG.

We check the unit axiom first: we have

$\forall \alpha_g \in k$. $\left(\sum_{g \in G} \alpha_g g\right) \cdot (e) = \sum_{g \in G} (\alpha_g \cdot 1)(g \cdot e) = \sum_{g \in G} \alpha_g g$ and similarly $e \cdot \sum_{g \in G} \alpha_g g = \sum_{g \in G} \alpha_g g$

the elt $e = 1 \cdot e \in kG$ is the unit of the mult in kG.

Associativity: Take three elts, say $x = \sum \alpha_g g$, $y = \sum \beta_h h$, $z = \sum \gamma_\ell \cdot \ell$, in kG.

Then
$$(xy)z = \left[\sum_{g,h} (\alpha_g \beta_h) gh\right] \cdot z = \sum_{g,h,\ell} \left[\underline{(\alpha_g \beta_h)(\gamma_\ell)}\right] \cdot \left[\underline{(gh)\ell}\right]$$

$$x(yz) = x \cdot \left[\sum_{h,\ell} (\beta_h \gamma_\ell)(h\ell)\right] = \sum_{g,h,\ell} \left[\underline{\alpha_g (\beta_h \cdot \gamma_\ell)}\right] \left[\underline{g(h\ell)}\right]$$

Since mult is associative in $\underline{k}$ and in $\underline{G}$, the right sides are equal, so

$(xy)z = x(yz)$. In other words, kG inherits associativity from k and G.

Bilinearity: similar proof; E.X. □

Note: The multiplication $(*)$ can be equivalently be given by

"define $g \cdot h = gh$ $\forall g, h \in G$, and extend bilinearly".

$(1 \cdot g)(1 \cdot h)$ ↘ def on the basis elts

2. Path algebras of quivers

Def. (quivers) A quiver is directed graph $Q=(Q_0, Q_1)$ where Q_0 is the vertex set and Q_1 is the directed edges or arrows. For each arrow $\alpha: a \to b$ in Q_1 ($a, b \in Q_0$), we say α has source a and target b, and we write $s(\alpha)=a$ and $t(\alpha)=b$.

Def. (paths/stationary paths) A path on a quiver Q is a sequence $p = \alpha_r \cdots \alpha_3 \alpha_2 \alpha_1$ of r arrows in Q s.t. $t(\alpha_i) = s(\alpha_{i+1})$ $\forall\, 1 \le i \le r-1$. We define the source of p to be $s(p) := s(\alpha_1)$ and the target of p to be $t(p) := t(\alpha_r)$.

For each vertex $a \in Q_0$, we define/allow a stationary path at a, denoted e_a, which just "stays at a".

Eg. Q:
$$\begin{tikzcd} c & b \arrow[l, "\beta"'] \arrow[loop below, "\gamma"] & a \arrow[l, "\alpha"'] \end{tikzcd}$$
$\rightsquigarrow$ stationary paths e_a, e_b, e_c. $\gamma \neq e_b$.

α, β are arrows, $\beta\alpha \overset{= \beta e_b \alpha}{}$ is a path,

$\alpha\beta$ is not a path / does not make sense.

Def (path algebras)

Let $Q = (Q_0, Q_1)$ be a quiver. The path algebra of Q is the free vector space kP on the set P of all paths on Q. To define multiplication on kP, we define

$$p_1 \cdot p_2 = \begin{cases} p_1 p_2 & \text{if } s(p_1) = t(p_2) \\ 0 & \text{otherwise} \end{cases}$$

and extend bilinearly.

$\alpha e_b = 0$.

Eg. For $Q: c \xleftarrow{\beta} b \xleftarrow{\alpha} a$, $\beta \cdot \alpha = (\beta\alpha)$, $\alpha \cdot \alpha = 0$, $\alpha \cdot \beta = 0$, $\alpha \cdot e_a = \alpha = e_b \alpha$,

$$(2\alpha + 3\beta) \cdot (5\beta - e_b) = 10 \cdot 0 + 15 \cdot 0 - 2 \cdot 0 - 3\beta = -3\beta.$$

Note: · unit : kP does have a unit, namely $\sum_{a\in Q_0} e_a$. (if $|Q_0| < \infty$, as we'll always assume.)

Pf: $\forall p \in P$, $\quad p\cdot \sum_{a\in Q_0} e_a = p\cdot e_{s(p)} + \sum_{\substack{a\in Q_0 \\ a\neq s(p)}} p\cdot e_a = p + \sum 0 = p$.

It follows that $\quad x\cdot \sum_{a\in Q_0} e_a = x \quad \forall x \in kP$.

Similarly $\quad \left(\sum_{a\in Q} e_a\right)\cdot x = x \quad \forall x \in kP$.

Therefore $\sum_{\omega\in Q_0} e_a$ is the unit of the path algebra.

· often we denote the path algebra as kQ rather than kP.

· kQ is finite dimensional, i.e., P is finite, iff Q has no "cycles" or "loops".

Next time:

more examples of alg homs & isos.