AA 2. Lecture 7.

k: a field

01. 26. 2022.

Last time:

- $k \hookrightarrow$ every k algebra A, $\lambda \mapsto \lambda \cdot 1_A$.
- alg. hom. $\quad \mathrm{End}_k(V) \cong M_n(k)$ if $\dim_k V = n$.
- quotient algebras: alg. A, two-sided ideal I $\Big\} \to$ A/I is an algebra w/ its operations naturally inherited from A.

Useful principle: to prove facts about algebras, build on facts from gp theory or linear algebra.

Today:

- review of well-definedness issues, proof about A/I
- algebra isomorphism theorems

1. Verify A/I is an alg.

Setup: A: k-algebra, I: two-sided ideal of A.

To check that A/I is an algebra under the inherited operations (Lecture 6):

· Recall from gp theory that $(A/I, +)$ is an abelian gp.

In particular, $+$ here is well-defined. $(a+I)+(b+I) = (a+b)+I$

i.e., If $a+I = a'+I$ and $b+I = b'+I$. Then $a-a' \in I$, $b-b' \in I$,

hence $(a+b)-(a'+b') = (a-a')+(b-b') \in I$,

So $(a+b)+I = (a'+b')+I$.

· Next, we check that the declared scaling operation is well-defined:

if $a+I = a'+I$ and $c \in k$, then $a-a' \in I$.

Then $ca - ca' = c \cdot (a-a') \in I$ since $a-a' \in I$ and I is an ideal

So $ca + I = ca' + I$, i.e. $c \cdot (a+I) = c \cdot (a'+I)$, so the scaling is well-defined.

· Well-definedness of mult: if $a+I = a'+I$, $b+I = b'+I$. then

$a-a' \in I$, $b-b' \in I$, so $(ab - a'b') = ab - ab' + ab' - a'b' = a\underbrace{(b-b')}_{I} + \underbrace{(a-a')}_{I} b'$.

Since $b-b' \in I$ and I is a left ideal, $a-a' \in I$ and I is a right ideal, the two summands are in I,

so $ab - a'b' \in I$. therefore $ab + I = a'b' + I$, as desired.

- Next, we need to check that the above operations satisfy the algebra axioms, making A/I an algebra.

↓

These axioms all follow from the corresponding properties for A since the operations in A/I are inherited from A.

Eg. (one condition for bilinearity of mult)

need $(a+I)\left[(b+I)+(c+I)\right] = (a+I)(b+I)+(a+I)(c+I)$ $\forall a,b,c \in A$

Pf: LHS $= (a+I)\left[(b+c)+I\right] = a(b+c)+I$

RHS $= (ab+I)+(ac+I) = (ab+ac)+I$.

Since A is an algebra, $a(b+c) = ab+ac$, so LHS = RHS. $\square$

Ex: Check the other axioms in similar routine ways.

2. Isomorphism Theorems.

Remark: Algebras are rings, and the following iso. thms are exactly copies of the ring iso. theorems with "rings" replaced with "algebras".

The 1st Iso Thm Let $\varphi: A \to B$ be an algebra hom. Then

(1) $\ker\varphi := \{a \in A : \varphi(a) = 0\}$ is a two-sided ideal of A,

and $\mathrm{im}\,\varphi := \{\varphi(a) : a \in A\}$ is a subalgebra of A.

(2) There is a well-defined algebra isomorphism $\bar\varphi : A/\ker\varphi \to \mathrm{Im}\,\varphi$ given by

$$\bar\varphi(a + \ker\varphi) = \varphi(a) \qquad \forall a \in A.$$

Pf: (1) HW. Example ingredient: $\cdot$ $\ker\varphi$ absorbs A on the left ($A \cdot \ker\varphi \subseteq \ker\varphi$)

since $\forall a \in A, x \in \ker\varphi$, $\varphi(ax) = \varphi(a)\varphi(x) = \varphi(a)\cdot 0 = 0$, hence $a \cdot x \in \ker\varphi$.

(2). ① Well-definedness : Suppose $a + \ker\varphi = a' + \ker\varphi$ for some $a, a' \in A$. Then $a - a' \in \ker\varphi$

$$\varphi(a) - \varphi(a') = \varphi(a - a') = 0.$$

So $\varphi(a) = \varphi(a')$. Therefore $\bar{\varphi}$ is well-defined. (or, you can recall this from gp theory)

② $\bar{\varphi}$ is an alg. hom. : · recall that $\bar{\varphi}$ is a gp hom, so it suffices to show that $\bar{\varphi}$ respects scaling, mult and sends unit to unit.
* * *

$\boxed{*}$ $\bar{\varphi}$ respects scaling: $\forall c \in k,\ a \in A$

$$\bar{\varphi}(c \cdot (a + I)) = \bar{\varphi}(c \cdot a + I) = \varphi(c \cdot a) \overset{\varphi \text{ is linear}}{=} c \cdot \varphi(a) = c \cdot \bar{\varphi}(a + I). \checkmark$$

✓ $\boxed{*}$ E.X.

✓ $\boxed{*}$ $\bar{\varphi}(1_A + \ker\varphi) = \varphi(1_A) = 1_B$ since φ is an alg. hom.

③ $\bar{\varphi}$ is bijective : recall from gp theory that $\bar{\varphi}$ is a bijective map.

By ①, ② and ③, we may conclude that $\bar{\varphi}$ is an alg. iso. □

The 2nd Iso Thm: Let A be a k-algebra. Let S be a subalgebra of A and let I be a two-sided ideal of A. Then

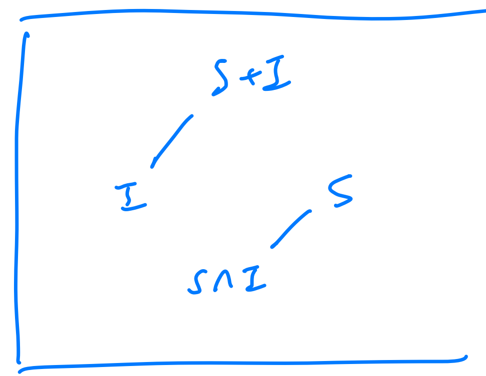
(1) $S+I$ is a subalgebra of A, and $I \cap S$ is a two-sided ideal of S.

(2). There is an algebra iso $\quad S/_{S\cap I} \cong {S+I}/_{I}$.

Pf: HW. For (2), construct a natural alg. hom

$$\varphi : S \longrightarrow {S+I}/_{I}$$

that is surj. (so $\operatorname{Im}\varphi = {S+I}/_{I}$) and $\ker\varphi = S\cap I$, then invoke the 1st Iso Theorem.



The 3rd Iso Thm. Let I and J be two-sided ideals of an algebra A, with $J \subseteq I$.

Then (1) the set $I/J = \{ i+J : i \in I \}$ is a two-sided ideal of A/J;

(2) there is an alg. iso $\dfrac{A/J}{I/J} \cong A/I$.

Pf: E.X.

The 4th Iso / Correspondence Thm Let A be an algebra and I a two-sided ideal of A.

Then there is an order ($\subseteq$)-preserving bijection

$$\Psi : \left\{ \begin{array}{c} \text{subalgebras of } A \\ \text{containing } I \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{subalgebras of} \\ R/I \end{array} \right\}.$$

given by $S \longmapsto S/I$ for every alg. S in the domain.

Next time: · group algebras · path algebras