AA2. Lecture 6. 01. 24. 2022.

Last time: examples of algebras:

k/k, $\mathbb{C}/\mathbb{R}$, the matrix algebra $M_n(k)$

polynomial algebras $k[x_1, x_2, \cdots, x_n]$, endomorphism algebras $\mathrm{End}_k(V)$

Note: To prove equality of two maps $f, g \in \mathrm{End}_k(V)$ is to prove that $f(v) = g(v)$ $\forall v \in V$. e.g.: to prove $(c\cdot f)g = c\cdot(fg)$, note that $\forall v \in V$, $[(c\cdot f)g](v) = (c\cdot f)(g(v)) = c(f(g(v))) = c\cdot(fg(v))$
$= [c\cdot(fg)](v)$.

Today:
- "k embedds into every k-algebra"
- algebra homomorphisms
- factor / quotient algebras

1. k embeds into k-algebras

Let k be a field.

Let A be a k-algebra. Consider the map $\iota : k \to A$, $\lambda \mapsto \lambda \cdot 1_A$.

<u>Note:</u> • (linearity) The map ι is linear.

ι is lin. $\Leftarrow \begin{cases} ① \ \forall \lambda, \lambda' \in k, \ \iota(\lambda + \lambda') = (\lambda + \lambda') \cdot 1_A \overset{\text{v.s axiom}}{=\!=} \lambda \cdot 1_A + \lambda' \cdot 1_A = \iota(\lambda) + \iota(\lambda') \\ ② \ \text{Ex:} \ \forall c \in k, \lambda \in k \quad \iota(c \cdot \lambda) = c \cdot \iota(\lambda) \end{cases}$

• (injectivity) The map ι is inj; equivalently, $\ker \iota = \{0\}$ HW.

• We have $\iota(1_k) = 1_k \cdot 1_A \overset{\text{v.s. axiom}}{=} 1_A$

and $\iota(\lambda)\iota(\lambda') = (\lambda \cdot 1_A)(\lambda' \cdot 1_A) = (\lambda\lambda') \cdot (1_A 1_A) = (\lambda\lambda') \cdot 1_A = \iota(\lambda\lambda')$

$\forall \lambda, \lambda' \in A$.

Later: The above facts imply that ι is an <u>inj.</u> <u>algebra homomorphism</u> ("embedding") from k to A.

Point: $\iota(k) = \{\iota(\lambda) : \lambda \in k\}$ lives as a "copy" of k in A.

In particular, scaling can be achieved by left and right mult. by A in the sense that $\forall \lambda \in k$, $a \in A$,

$$\lambda \cdot a = \lambda \cdot (1_A a) = (\lambda 1_A) a, \quad \text{and similarly } \lambda \cdot a = a(\lambda \cdot 1_A)$$

Consequences: Given a left or right ideal I of A, since $AI \subseteq I$ or $IA \subseteq I$, I must be closed under scaling and hence automatically a subspace of A. (see also Lemma 1.20)

2. Homomorphisms and related notions

Def. Let A and B be k-algebras. A map $f: A \to B$ is called an <u>algebra homomorphism</u> if it's a <u>linear</u> map s.t. <u>$f(1_A) = 1_B$</u> and <u>$f(aa') = f(a) f(a')$ $\forall a, a' \in A$</u>. An <u>algebra isomorphism</u> is a bijective alg. hom.

Example. $\mathrm{End}_k(V) \cong M_n(k)$ as k-algebras if $\dim_k V = n$.

Pf (sketch): Fix a basis $\beta = \{v_1, \dots, v_n\}$ of V.

Consider the map $\varphi : \underbrace{\mathrm{End}_k(V)}_{A} \to \underbrace{M_n(k)}_{B}$

with $\varphi(f) = [f]_\beta$, the matrix of f w.r.t. β, for all $f \in \mathrm{End}_k(V)$.

↓ Lecture 4 i th col.

$$[f]_\beta = \left[\cdots \; [f(v_i)]_\beta \; \cdots \right]$$

Why is φ inj/surj?

Note: · Recall (or Ex.) that φ is a linear map, and it's bijective.

(So φ is a vector space isomorphism.)

· It remains to show that $\varphi(\underbrace{\mathrm{Id}_V}_{1_A}) = \underbrace{I_n}_{1_B}$ and $\varphi(fg) = \varphi(f)\varphi(g)$ $\forall f, g \in \mathrm{End}_k(V)$

$[Id_V]_\beta = I_n$: $[Id_V]_\beta = \left[[Id_V(v_1)]_\beta \; \cdots \; [Id_V(v_n)]_\beta \right]$

$$= \left[[v_1]_\beta \; \cdots \; [v_n]_\beta \right]$$

$$= \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix} = I_n \quad \checkmark$$

$\varphi(fg) = \varphi(f)\varphi(g)$: we need $[fg]_\beta = [f]_\beta [g]_\beta$ $\forall$ lin. maps $f, g : V \to V$.

This holds by the property of matrix mult. from linear algebra!

We have proved that φ is an alg. iso, so $\mathrm{End}_k(V) \cong M_n(k)$ as algebras. $\square$

3. Factor / quotient algebras (by two-sided ideals)

Prop: Let A be a k-algebra and let I be a two-sided ideal of A.

Then the quotient group A/I (where the elts are cosets $a+I$, $a\in A$, with $a+I = b+I \Leftrightarrow a-b\in I$)

forms a k-algebra under the following well-defined operations

inherited from A:

$$\left.\begin{array}{ll} + : & (a+I)+(b+I) = (a+b)+I \\ \text{scaling}: & \lambda\cdot(a+I) = (\lambda a)+I \\ \text{mult}: & (a+I)(b+I) = ab+I \end{array}\right\} \forall a, b\in I,\ \lambda\in k.$$

We call A/I a factor or quotient algebra (of A).

Rmks:

- When dealing with algebras, we may often invoke results from gp theory and linear algebra to reduce work.

e.g.:

- We invoked lin. alg. when proving "$\mathrm{End}_k(V) \cong M_n(k)$".
- if we assume ring theory, we'd know the three operations in A/I are well-defined and makes A/I a ring, so it remains to check only the alg. axioms involving scaling.

Next time:

- reminders about well-definedness; a pf assuming only gp theory;
- iso. theorems.