AA 2. Lecture 5.

01. 21. 2022.

Last time:

- Bases control vector spaces.
- Def of algebras over fields: vector spaces equipped with unital, associative, and bilinear multiplication.
- Ideals and/v.s. subalgebras of an algebra A

  ideals: subgps absorbing mult. by A on the suitable side(s)

  Subalgebras: Subsets containing 1_A and closed under lin comb & mult.

Today:

- examples of algebras
- algebra homs, "$\mathrm{End}_k(V) \cong M_n(k)$ if $\dim_k V = n$"

1. Examples of algebras

Let k be a field as usual.

(1). k/k : k is an algebra over k.

- We already know k is a k-vector space.
- has mult? Sure. the mult in k.
- unital? Yes, k has a unit since it's a field.
- mult is associative? Yes, by field axioms.
- mult. is bilinear?

$\forall \lambda \in k$, $a, b \in k$

$$\boxed{\begin{aligned}(a+a')\cdot b &= ab + a'b \\ a\cdot(b+b') &= ab + ab'\end{aligned}} \qquad \boxed{\begin{aligned}(\lambda\cdot a)b &= \lambda\cdot(ab) \\ a(\lambda\cdot b) &= \lambda\cdot(ab)\end{aligned}}$$

hold by field axioms

follows from associativity, comm., and the scalar action being mult:

$\lambda\cdot a = \lambda a$

(2) $\mathbb{C}/\mathbb{R}$, similar to (1) : $\mathbb{C}$ is an algebra over $\mathbb{R}$.

Ex: Carefully check the necessary axioms.

(3) $M_n(k) / k$ ← note: this algebra is commutative only when $n = 1$.

$\{ n \times n$ matrices w/ entries from $k \}$.

- $M_n(k)$ is a vector space. (in particular, one can scale matrices)
- has mult? Yes, can multiply $n \times n$ matrices together,
- unit? Yes, $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is the unit: $A I_n = A = I_n A$ $\forall A \in M_n(k)$
- is matrix mult associative? Yes by linear algebra.
- ------- bilinear? $(A+A')B = AB + A'B$, $(\lambda \cdot A) B = \lambda \cdot (AB)$
  $A(B+B') = AB + AB'$, $A(\lambda \cdot B) = \lambda \cdot (AB)$
  $\forall A, A', B, B', \lambda$ by lin algebra.

Q: Is $GL_n(k)$ a subalgebra of $M_n(k)$?

$GL_n(k) = \{$ invertible $n \times n$ matrices over $k \}$

- $1 = I_n \in GL_n(k)$? Yes.
- closure under lin. comb? No! → So $GL_n(k)$ is not a subalgebra of $M_n(k)$.
- closure ... mult? Yes

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q' (Lex): Is $B = GL_n(k) \cup \{0\}$ a subalgebra of $M_n(k)$?

No, B is not closed under lin. comb.

$$\underset{\in B}{\begin{bmatrix}1&2\\3&4\end{bmatrix}} + \underset{\in B}{\begin{bmatrix}0&-1\\-2&-3\end{bmatrix}} = \underset{\notin B}{\begin{bmatrix}1&1\\1&1\end{bmatrix}}.$$

(4) polynomial algebras

$k[t]$, the univariate polynomial space over k, is a k-algebra.

variable

- have addition, scaling and mult? Yes.
- the axioms? Ex: They do hold. (1 is the constant poly $f = 1$.)

Note: . The subspace $B = \mathrm{Span}_k\{t^i : i \geq 1\}$ = {all polynomials with no constant term} $= \langle t \rangle$.

Is an ideal (left, right, two-sided → doesn't matter since $k[t]$ is commutative).

- The subspace $B' = \mathrm{Span}_k\{t^{2i} : i \geq 0\}$ = {all even polynomials in $k[t]$}

is not an ideal but is a subalgebra.

($(3 - 2t^2) \cdot (t^4 - 3t^2 + 5)$ = lin. comb. of x^{2i}, $i \geq 0$)

(5) Endomorphism algebras

Next time: $\mathrm{End}_k(V) \cong M_n(k)$ as algebras.

Let V be a vector space over k. Suppose further that V is fin. dim; say $\dim_k V = n$.

Claim: The vector space $\mathrm{End}_k(V) = \{k\text{-linear maps } f: V \to V\}$ "endomorphisms" form a k-algebra under composition (as the multiplication).

Ex: check that $\mathrm{End}_k(V)$ does form a v.s. under the usual addition and scaling of functions.

- unit? Yes, the identity map is the unit. $id_V : V \to V$, $x \mapsto x$ $\forall x \in V$
- associativity? $(f \circ g) \circ h = f \circ (g \circ h)$ by how functions compose.
- bilinearity? Ex: check the four axioms.

e.g. $f \circ (g+h) = f \circ g + f \circ h$

just need to check that $[f \circ (g+h)](x) = [f \circ g + f \circ h](x)$ $\forall x \in V$.