AA2. Lecture 4.

01. 19. 2022.

Last time: review for vector spaces, linear maps, bases, dimensions, ...

Today:

- more on basis
- def of algebras over fields and related basic notions (substructures and morphisms)

1. Bases

Key point to recall: The behavior of a v.s. is often "controlled" by the behavior of any chosen basis of it.

Eg. Let V be a k-vec. space and let B be a basis of V.

(1) Decomposition: Every vector $u \in V$ has a <u>unique</u> decomposition as a linear comb.
$$u = \sum_{w \in B} c_w \cdot w \quad \text{where } c_w = 0 \text{ for all but finitely many elts } w \in B.$$

(2) linear map: If $f : V \to V'$ is a linear map. Then the image of f is controlled by the images of the basis elts in the sense that
$$u = \sum c_w \cdot w \implies \underbrace{f(u)}_{\text{new}} = f\left(\sum c_w \cdot w\right) = \sum c_w \cdot \underbrace{f(w)}_{\text{"old" - in } \{f(w) : w \in B\}}$$

(3) Matrix of a linear map: ← more on this later with examples

Let $f: V \to V'$ be a linear map. (For simplicity) suppose V and V' are finite dimensional, so that B is finite and V' has a finite basis B'.

Say $B = \{w_1, w_2, \cdots, w_n\}$ and $B' = \{w_1', w_2', \cdots, w_m'\}$.

Then the matrix of f with respect to B and B' is the $m \times n$ matrix

$$[f]_B^{B'} := \begin{array}{c} \\ \left[\begin{array}{c:c:c} \cdots & [f(w_i)]_{B'} & \cdots \end{array} \right] \\ \text{ith column} \end{array},$$

where the ith col is $[f(w_i)]_{B'}$, the coordinate vector of $f(w_i)$ relative to B' (i.e., the vector $\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$ s.t. $f(w_i) = \sum_{i=1}^{m} d_i w_i'$).

Recall that $[f]_B^{B'}$ has the property that

$$[f(u)]_{B'} = [f]_B^{B'} \cdot [u]_B \quad \forall u \in V.$$

2. Basic notions for algebras

Let k be a field for the rest of today.

Def: (k-algebras) An <u>algebra over k</u> or a <u>k-algebra</u> is a k-vector space $(A, +, \cdot)$ equipped with an associative, bilinear and unital multiplication $m : V \times V \to V$ (often written as juxtaposition).

scaling

Here,
- associativity means that $(ab)c = a(bc)$ $\forall a, b, c \in A$
- bilinearity means that $(a+a')b = ab + a'b$, $a(b+b') = ab + ab'$, $(\lambda a)b = \lambda(ab)$, $a(\lambda b) = \lambda(ab)$ $\forall \lambda \in k$ $\forall a, b, a', b' \in A$.
- unitality means that the mult. has a unit.

Ex: Convince yourself that a k-alg is just a k-vec. space that is also a ring and satisfies the yellow-shaded conditions.

Rmks: Let A be a k-algebra.

- The dimension of A is just defined to be the dimension of A as a vector space.

- We call A commutative if its mult. is commutative, i.e.,, if $ab = ba$ $\forall a, b \in A$.

- The fact that A is an abelian gp and satisfies the blue-shaded conditions implies that A is a ring, so $\{k\text{-algebras}\} \subseteq \{\text{rings}\}$ and $\{\text{comm. } k\text{-alg.}\} \subseteq \{\text{Comm. rings}\}$. But k-algebras have more structures: it has scaling which are compatible with $+$ (to make it a k-v.s.) and compatible with mult (the yellow-shaded conditions).

Def. (ideals) Let A be a k-algebra. A left ideal of A is a left ideal of A in the ring sense, i.e., it's a subgp I of A s.t. $a \cdot i \in I \ \forall\, a \in A,\ i \in I$. ($A \cdot I \subseteq I$)

Similarly, a right ideal of A is just a subgp I of A s.t. $I \cdot A \subseteq I$.

A two-sided ideal of A is a subgp I of A s.t. $A \cdot I \subseteq I$ and $I \cdot A \subseteq I$.

Ex: Use the above definitions to show that any ideal of A is automatically a subspace of A.

Def. (Subalgebras) Let A be a k-algebra. A subalgebra of A is a subset B of A that contains 1_A and is closed under linear combinations (+ and scaling) multiplication (i.e. $B \cdot B \subseteq B$).

Rmks:

- Ex: Show that a subalgeba of A is just a subset of A containing 1_A that is an algebra itself under the addition, scaling, and mult. operations inherited from A.

- Note some key differences between the definitions of ideals and subalgebras:
  - subalgebras are required to contain 1_A. ideals are not (see HW.)
  - for a subset $B \subseteq A$ to be an ideal, we need $\underline{\underline{A}} \cdot B \subseteq B$ or $B \cdot \underline{\underline{A}} \subseteq B$.

    . . . . a subalgebra, we need $B \cdot B \subseteq B$.

Next time: homs ; examples of algebras.