AA2. Lecture 39.

04. 25. 2022.

Last time:

- A.W. decomp. of s.s. gp algebras of finite gps
- Pf of Maschke's Thm: the "only if" direct, a lemma for the "if" direction.

$$\left.\begin{array}{l} kG \text{ s.s} \\ G \text{ finite} \end{array}\right. \Rightarrow \operatorname{char} k \nmid |G|$$

Today:

- prove the if direct.
- $\#$(simples of dim 1 for s.s. kG)

Next time: course review

1. Finishing the proof of Maschke's Thm

Where we are: It remains to prove that for a finite gp G, kG is a s.s. algebra if $\mathrm{char}(k) \nmid |G|$.

- We will prove kG is s.s by proving that the regular module kG is completely reducible, by showing that any submodule has a complement C in kG.
  ($kG = W \oplus C$)
- To find C, we recall from the lemma last time that

  if $j: N \to M$ and $\pi: M \to N'$ are A-module homs for an alg. A with $\pi \circ j$ being an iso, then $M = \operatorname{im} j \oplus \ker \pi$. We note that with $N \xrightarrow{j} M \xrightarrow{\pi} N'$

  $N = W$, $M = kG$ and $j = \kappa_W : W \to kG$ the natural embedding, we get

$W = \operatorname{im} j \oplus \ker\pi = W \oplus \ker\pi$.

Thus, it remains to find an kG-module N' and an kG-mod hom

$\pi : kG \to N'$ s.t. $\pi \circ j$ is an iso.

Claim: Taking $N' = W$, taking $kG = W \oplus V$ to be any v.s. decomp. of kG.

and $\pi : kG \to W$ to be the map s.t.

$$\pi(m) = \frac{1}{|G|} \sum_{g \in G} g \cdot p(g^{-1} m) \qquad \forall m \in kG$$

where $p : kG = W \oplus V \to W$ is the proj. map w.r.t to the v.s. decomp

$kG = W \oplus V$ (and then taking $C = \ker\pi$) works.

Pf of the claim: $\quad \pi(m) = \frac{1}{|G|} \sum_{g\in G} g\cdot p(g^{-1}\cdot m)$

We need to prove that $\underbrace{\pi \text{ is an } kG\text{-mod hom}}_{(1)}$ with $\underbrace{\pi \circ j = Id_W}_{(2)}$.

certainly an iso.

(2): $\forall w \in W$,

$g^{-1}\cdot w \in W$, since W is a submod

$$\pi \circ j\,(w) = \pi(w) = \frac{1}{|G|} \sum_{g\in G} g\cdot p\cdot(g^{-1}\cdot \underset{\substack{\uparrow \\ W}}{\underline{w}}) = \frac{1}{|G|} \sum_{g\in G} g\cdot g^{-1}\cdot w$$

$$= \frac{1}{|G|} \sum_{g\in G} w = \frac{1}{|G|}\cdot |G|\cdot w = w.$$

(1) linearity of π: routine, follows from the linearity of $g\cdot -$, p, $g^{-1}\cdot -$. $\forall g\in G$.

It remains to show that π respect the kG-actions, i.e. that

$$\pi(h\cdot m) = h\cdot \pi(m) \qquad \forall m \in kG,\ h\in G.$$

"$\pi(h.m) = h.\pi(m)$": $\forall m \in kG,\ h \in G.$

$$\pi(m) = \frac{1}{|G|} \sum_{g \in G} g.p(g^{-1}.m)$$

$$h.\pi(m) = h.\frac{1}{|G|} \sum_{g \in G} g.p.g^{-1}(m) = \frac{1}{|G|} \cdot \sum_{g \in G} \underline{hg}.p(g^{-1}.m)$$

$g^{-1} = x^{-1}h$, $g = h^{-1}x$ $\Leftarrow$ $x = hg$

$$\overset{x=hg}{=\!=} \frac{1}{|G|} \sum_{x \in G} x.p(x^{-1}h.m)$$

$$\pi(h.m) = \frac{1}{|G|} \sum_{g \in G} g.p\left(g^{-1}(h.m)\right) \overset{x=g}{=\!=} \frac{1}{|G|} \sum_{x \in G} x.p(g^{-1}h.m)$$

It follows that $h.\pi(m) = \pi(h.m)$. so π is a kG-mod hom,

and we are done. $\square$

2. # (Simples of dimension 1)

Let G be a finite gp. Suppose $k=\bar{k}$ and $\operatorname{char}(k)=0$ (e.g. $k=\mathbb{C}$), so that kG is s.s. and has A.W. decomp $kG \cong M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k)$ as algebras where $r = \#\text{simples of } kG/\text{iso} = \#\text{conj classes of } G$ and where we may assume $n_1 = 1$, corresponding to the trivial module of dim 1.

One more numerical fact: recall/note from gp theory the commutator subgp of G is the subgp G' generated by elts of the form $[xy] = xyx^{-1}y^{-1}$ $x, y \in G$.

Fact/Prop: Let G, k, n_i $(1 \le i \le r)$ be as above. Let $N = \left\{ n_i \;\middle|\; \begin{array}{l} 1 \le i \le r \\ n_i = 1 \end{array} \right\}$.

Then $|N| = |G/G'| = |G|/|G'|$. In particular, $|N|$ divides $|G|$.

Example:

HW 6.9: Is there a finite G for which $\mathbb{C}G$ has the following A.W. decomp?

(a) $M_3(\mathbb{C})$. No, there should be at least one $M_1(\mathbb{C})$ comp.

(b) $\mathbb{C} \times M_2(\mathbb{C})$ No. think dim. and comm. (G is abelian $\Updownarrow$ "$n_i = 1 \ \forall i$")

(c) $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ Yes. S_3.

(d) $\mathbb{C} \times \mathbb{C} \times M_3(\mathbb{C})$ No, same reasoning as (b) or use the fact that $|N| \,\big|\, |G|$.