AA2. Lecture 38.

04. 22. 2022.

Last time:

- A.W. decompositions of s.s. quotients of $k[x]$, s.s. path algebra, s.s group algebras.

  "$Q_1 = \phi$" $\Updownarrow$ s.s. path algebra

- Maschke's Thm: G is a finite gp $\Rightarrow$ kG is s.s. iff $\mathrm{char}(k) \nmid |G|$.

Today:

- A closer look at the A.W. decomp of $\mathbb{C}G$, G finite gp.

- Proof of Maschke's Thm:

  — The "only if" proof

  — Start the "if" proof.

1. Finite dimensional group algebras

Generalizing our discussion for the finite gps S_3 and S_4, we have:

Thm (Thm 6.4) Let G be a finite gp. Then $\mathbb{C}G$ is s.s and therefore has A.W. decomposition $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$.

Moreover, (a) The alg. $\mathbb{C}G$ has exactly r simple modules up to iso; the dimensions of these modules are $n_1, \ldots, n_r$.

(b) We have $|G| = \sum_{i=1}^{r} n_i^2$.

(c) The gp G is abelian iff all simple G-modules ($\mathbb{C}G$-modules) have dim 1.

Another useful fact: · (Thm 6.12) The number r coincides w/ the number of conj. classes in G

Pf:

(a) Corollary of Maschke's Thm and (the detailed version of) the A.W. thm.

(b) Follows from the A.W. decomp. by taking dimensions on both sides.

(c) G is abelian $\Leftrightarrow$ $\mathbb{C}G$ is comm. $\overset{[3]}{\Leftrightarrow}$ $n_i = 1 \ \forall i$.

2. Proof of Maschke's Thm ($|G| < \infty \Rightarrow kG$ is s.s. iff $\mathrm{char}(k) \nmid |G|$)

Pf of the only if direction: Let G be a finite gp. Suppose kG is s.s.

Key object: the elt $w := \sum_{g \in G} g \in kG$. $h \cdot w = w \ \forall h \in G$, so w is fixed by kG.

Note: · $\forall h \in G$, we have $h \cdot w = \sum_{g \in G} \underline{hg} = \sum_{x \in G} x = w$ since the map $l_h : G \to G$, $g \mapsto h \cdot g$ is a bijection

· Consequently, the subspace $U := \mathrm{Span}\{w\}$ is a submodule of kG,

and $w \cdot w = \left(\sum_{g \in G} g\right) \cdot w = \sum_{g \in G} g \cdot w = \sum_{g \in G} w = |G| w$.

(The proof that $\mathrm{char}(k) \nmid |G|$) Since kG is s.s., kG is completely reducible, so that U has a complement, say C, in kG, that is, we have $kG = U \oplus C$.

($kG = U \oplus C$) In particular, we have $\underbrace{1_{kG}}_{\text{"}e_G\text{"}} = \lambda w + c$ for some $\lambda \in k$ and $c \in C$.

Contradiction

Note that $\lambda \neq 0$: otherwise $1_{kG} = c \in C$ so $g = g \cdot \underset{C}{1} \in C$ $\forall g \in G \Rightarrow kG = C$

But then $w = w \cdot 1_{kG} = w(\lambda w + c) = \lambda w^2 + w \cdot c = \lambda |G| w + w \cdot c$

so $\underbrace{w \cdot c}_{C} = w - \lambda |G| w = \underbrace{(1 - \lambda |G|) w}_{U}$ $\in$ $C \cap U = 0$

It follows that $\lambda |G| = 1 \neq 0$, so in particular $|G| \neq 0$ in k,

so $\mathrm{char}(k) \nmid |G|$.

Pf of the if direction: Suppose $\mathrm{char}(k) \nmid |G|$.

Strategy: We will show kG is s.s. by showing it's completely reducible.

To this end, we will show that any submodule W of kG has a complement C; we will explicitly construct C using the following lemma.

A useful lemma: Let A be a k-alg. Let $j: N \to M$ and $\pi: M \to N'$ be A-module homs s.t. $\pi \circ j : N \to M \to N'$ is an iso, then $M = \mathrm{Im}\, j \oplus \ker \pi$.

(We'll take $N = W$, $M = kG$, $j = $ (the natural embedding $\iota_W: W \to kG$) and design π carefully with $\ker \pi = C$, so $kG = W \oplus C$.)

Pf of the Lemma: $\left(\begin{tikzcd} N \arrow[r, "j"] & M \arrow[r, "\pi"] & N' \arrow[ll, bend left, "m"] \end{tikzcd} \text{ iso} \Rightarrow M = \operatorname{im} j \oplus \ker \pi\right)$

- Recall $\operatorname{im} j$ and $\ker \pi$ are submodules of M.

should be $j(n)$ for some $n \in N$.

- $M = \operatorname{im} j + \ker \pi$: Take $m \in M$. we need to find $m_1 \in \operatorname{im} j$ and $m_2 \in \ker \pi$ s.t. $m = m_1 + m_2$. Claim: Taking $m_1 = j\left(\underline{\underline{(\pi \circ j)^{-1}(\pi(m))}}\right)$, i.e., taking m_1 to be $j(n)$ where n is the unique elt in N s.t. $(\pi \circ j)(n) = \pi(m)$, and taking $m_2 = m - j(n)$ works. Pf: Ex.

- $\operatorname{im} j \cap \ker \pi = \{0\}$: Let $x \in \operatorname{im} j \cap \ker \pi$. Then $x = j(y)$ for some $y \in N$ and $\pi(x) = 0$. Thus, $\pi \circ j(y) = \pi(x) = 0$. Since $\pi \circ j$ is an iso, it follows that $y = 0$, so it further follows that $x = j(y) = 0$.

It follows that $M = \operatorname{im} j \oplus \ker \pi$. $\square$

We'll finish the "if" proof next time!.