AA2. Lecture 37.

HW:
- deadline extended to this Friday
- next hw will be due next Friday

04. 20. 2022.

Last time: finished the pf of the A.W. thm.

Today:
- Some applications of the A.W. thm.
- Maschke's Thm: statement and first examples

1. Remarks on / Applications of the A.W. Thm

Remark: (Commutativity) By the A.W. Thm, if k is alg. closed ($k = \bar{k}$), then A is a s.s. k-algebra $\Leftrightarrow$ $A \cong M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k)$ for some $r \in \mathbb{Z}_{\geq 1}$, $n_1, \dots, n_r \in \mathbb{Z}_{\geq 1}$.

$M_n(k)$ is commutative iff $n = 1$.

If this is the case, we note that A is commutative iff $n_i = 1 \ \forall i$, i.e., (if $k = \bar{k}$) comm. s.s. algebras $\equiv$ direct products of k.

More generally, over an arbitrary field k, A is comm. and s.s iff A is a direct product of commutative division algebras over k $\left(\prod_{i=1}^{r} D_i = \prod_{i=1}^{r} M_{\underline{1}}(D_i) \right)$, which has to be a direct product of has to be a field containing k fields containing k.

What about our three main types of examples? (What do the A.W. decomps look like?)

(1) Path algebras

Recall that for an acyclic quiver $Q = (Q_0, Q_1)$, we proved using the theory of Jacobson radicals that $J(kQ) = \mathrm{Span}_k\{\text{all positive length paths on } Q\} = kQ^{\geq 0}$ and hence kQ is s.s iff $Q_1 = \phi$, i.e., iff Q has no arrows.

When this is the case, $\underbrace{kQ = \prod_{i \in Q_0} k}_{}$, which is commutative.

↓

the A.W. decomp of a s.s. path algebra.

(2) Polynomial rings

Recall: • $k[x]$ is not s.s.

• For $f \in k[x]$ w/ irr. decomp $f = f_1^{a_1} f_2^{a_2} \cdots f_r^{a_r}$, the algebra $k[x]/\langle f \rangle$ is s.s iff $a_i = 1 \ \forall i$. In this case ...

(1) if $k = \mathbb{C} = \bar{k}$, then all the irreducible polys. $f_1, \ldots, f_r$ has to be of degree 1 and of the form $(x - c_i)$ for some scalars $c_1, \ldots, c_r$. So

$$k[x]/\langle f \rangle = k[x]/\left\langle \prod_{i=1}^{r} (x - c_i) \right\rangle \overset{\text{Chinese Remainder Thm}}{=\!=\!=} \prod_{i=1}^{r} k[x]/\langle x - c_i \rangle$$

$\Downarrow$ Same conclusion

$$\| \text{ our remark} \to \text{Same conclusion}$$

$$\prod_{i=1}^{r} k$$

A.W. decomp of $k[x]/\langle f \rangle$

(2). if $k = \mathbb{R} \neq \bar{k}$, then (fact:) every irr. poly. over $\mathbb{R} = k$ has degree 1 or 2, (e.g. $x-1$, x^2+1), so we may assume (by reordering terms if necessary) that $f = \underbrace{(f_1 \cdot f_2 \cdots f_m)}_{\text{deg. 1 irr.}} \underbrace{(f_{m+1} \cdots f_r)}_{\text{deg-2 irr.}}$. It follows that

$$\underbrace{k[x]/\langle f\rangle = k[x]/\langle f_1\rangle \times \cdots \times k[x]/\langle f_m\rangle \times k[x]/\langle f_{m+1}\rangle \times \cdots \times k[x]/\langle f_r\rangle \xrightarrow{\text{Rmk 5.10.}} = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{m} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r-m}.}_{\text{A.W. decomp for s.s. quotients } \mathbb{R}[x]/\langle f\rangle \text{ of } \mathbb{R}[x].}$$

(3) Group algebras.

For finite groups, we have Maschke's Thm ...

2. Maschke's Thm (statement and examples)

Thm: (Thm 6.3). Let k be a field and let G be a finite group. Then the group algebra kG is s.s. iff the characterisez $\mathrm{char}(k)$ of k does not divide $|G|$. In particular, if $\mathrm{char}(k) = 0$ then kG is s.s for every finite group G.

Note:
- Fact: If $\mathrm{char}(k) = 0$ then #simple modules of kG up to iso = # conj. classes in G for any finite group G.
- Combining Maschke's Thm, the A.W. theorem and the above fact can help us find the A.W. decomp. of kG for finite G. / k with $\mathrm{char}(k) = 0$.

E.g. $k = \mathbb{C}$, $G = S_3$.
$\quad\ \ \overline{k}$ (with $\mathbb{C} = \overline{k}$)

- By Abstract Algebra 1, # conj. classes of S_3 = # cycle types for S_3 = 3

$(a)(b)(c)$, $(ab)(c)$, (abc)

So $\mathbb{C}S_3$ has 3 simple modules (up to iso),

so the A.W. decomp of $\mathbb{C}S_3$ should be $\mathbb{C}S_3 = M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times M_{n_3}(\mathbb{C})$.

- By dim. consideration, we should have $n_1^2 + n_2^2 + n_3^2 = |S_3| = 6$

$$\Downarrow$$

$n_1 = n_2 = 1 \quad n_3 = 2$ if we assume $n_1 \leq n_2 \leq n_3$

- Thus, $\mathbb{C}S_3$ must have two simple modules of dim. 1 and one simple of dim 2.

What are they?

makes sense for any group.

Simple 1: (the trivial module $k = \mathbb{C}$ on which $g \in G$ acts as 1 $\forall g \in G$.) $\overset{\text{"}}{\cong}$ $\text{Span}\{v_1+v_2+v_3\}$ $\overset{\text{"}}{\subseteq}$ k^3

the perm. module

dim-2 simple: $\text{"}\{av_1+bv_2+cv_3 \in k^3 \mid a+b+c=0\} = \text{Span}\{v_1-v_2, v_2-v_3\}$ $\overset{\text{"}}{\subseteq}$ k^3 (from HW.)

The third simple / second dim-1 simple: $V = k = \mathbb{C}$. with the action

$$g \cdot 1 = \text{sign}(g),$$

} the sign rep.

corresponding to the rep $\mathbb{C}G \to \text{End}_{\mathbb{C}}(V) = \mathbb{C}$,

$$g \mapsto \text{sign}(g)$$

Q: Can you find the A.W. decomp for $\mathbb{C}S_4$?

Next time: proof of Maschke's Thm.