AA2. Lecture 36.

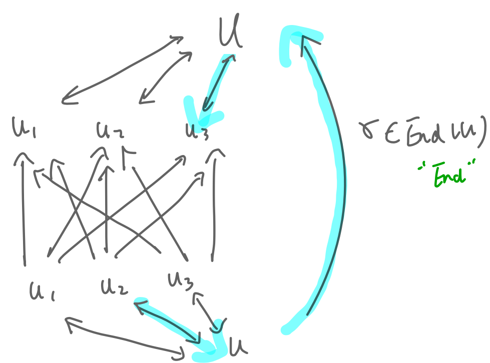
04. 18. 2022.

So far, we've proved the following towards the A.W. Theorem, for any algebra A:

(1) $A = \left(\mathrm{End}_A(A)\right)^{op}$

(2)

"matrices"
Λ.



If $U_1, \cdots, U_n$ are A-modules, then

$$\mathrm{End}_A(\oplus U_i) \cong \left\{ [\varphi_{ij}]_{i,j} \;\middle|\; \varphi_{ij} \in \mathrm{Hom}(U_j, U_i) \ \forall i,j \right\}$$

$$\parallel$$

$$\Lambda$$

Today: Putting together (1), (2) and Schur's Lemma to finish the proof.

1. Proof of the A.W. Thm.

Setup: A: s.s. k-algebra, decomposing into (as a regular module)

$$A = \left(\underset{\substack{\downarrow \\ \text{each} \cong S^{(1)}}}{S_1^{(1)} \oplus S_2^{(1)}} \oplus \cdots \oplus S_{n_1}^{(1)}\right) \oplus \left(S_1^{(2)} \oplus S_2^{(2)} \oplus \cdots \oplus S_{n_2}^{(2)}\right) \oplus \cdots \oplus \left(S_1^{(r)} \oplus \cdots \oplus S_{n_r}^{(r)}\right)$$

each $\cong S^{(2)}$ $\qquad$ each $\cong S^{(r)}$

where $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ is a full list of pairwise non-iso. simples in A.

Def: Let $B^{(i)} := \bigoplus_{j=1}^{n_i} S_j^{(i)}$ $\to$ the ith block.

Let $\tilde{D}_i := \mathrm{End}_A(S^{(i)}) \overset{\text{E.x.}}{\cong} \underline{\mathrm{Hom}_A(S_j^{(i)}, S_k^{(i)})}$ $\to$ a division algebra by Schur's Lemma.

Also recall from Schur's Lemma: $\underline{\mathrm{Hom}_A(S_j^{(i)}, S_k^{(i')})} = 0$ whenever $i \neq i'$ since $S^{(i)} \not\cong S^{(i')}$

where the "φ_{ij}"s come from in Λ.

Example: Say $A = \underbrace{\left(S_1^{(1)} \oplus S_2^{(1)}\right)}_{1,\,2} \oplus \underbrace{\left(S_1^{(2)}\right)}_{3} \oplus \underbrace{\left(S_1^{(3)}\right)}_{4}$

Then by (2),

$$\mathrm{End}_A(A) = \Lambda = \left\{ \left[\begin{array}{cc|c|c} \varphi^{(1)(1)}_{1\,1} & \varphi^{(1)(1)}_{1\,2} & \varphi^{(1)(2)}_{1\,1} & \varphi^{(1)(3)}_{1\,1} \\ \varphi^{(1)(1)}_{2\,1} & \varphi^{(1)(1)}_{2\,2} & \varphi^{(1)(2)}_{2\,1} & \varphi^{(1)(3)}_{2\,1} \\ \hline \varphi^{(2)(1)}_{1\,1} & \varphi^{(2)(1)}_{1\,2} & \varphi^{(2)(2)}_{1\,1} & \varphi^{(2)(3)}_{1\,1} \\ \hline \varphi^{(3)(1)}_{1\,1} & \varphi^{(3)(1)}_{1\,2} & \varphi^{(3)(2)}_{1\,1} & \varphi^{(3)(3)}_{1\,1} \end{array}\right] : \varphi^{(i)(i)}_{k\ j} \in \mathrm{Hom}(S^{(i)}_j, S^{(i)}_k) \right\}$$

by the last page

$$= \left\{ \left[\begin{array}{c|c|c} M_1 & 0 & 0 \\ \hline 0 & M_2 & 0 \\ \hline 0 & 0 & M_3 \end{array}\right] : M_i \text{ has entries from } \widetilde{D}_i \text{ and has size } n_i \times n_i \right\}$$

$M_i \in M_{n_i}(\widetilde{D}_i)$

E.X. $\rightarrow$ $\cong M_{n_1}(\widetilde{D}_1) \times M_{n_2}(\widetilde{D}_2) \times M_{n_3}(\widetilde{D}_3)$

More generally, a similar argument shows that

$$\mathrm{End}_A(A) \cong \prod_{i=1}^{r} M_{n_i}(\tilde{D}_i).$$

<u>Note</u>: If k is alg. closed and A is f.d. (which implies that each $S^{(i)}$ is f.d.), then by Schur's Lemma $\tilde{D}_i = k \ \forall i$.

<u>The endgame</u>: A s.s $\Rightarrow$

$$\begin{aligned} A &\cong (\mathrm{End}_A(A))^{op} \\ &\cong \left(\prod_i M_{n_i}(\tilde{D}_i)\right)^{op} \\ &= \prod_i \left(M_{n_i}(\tilde{D}_i)^{op}\right) \\ &\overset{\text{H.W.}}{\cong} \prod_i \left(M_{n_i}(\boxed{\tilde{D}_i^{op}})\right) = \prod_i M_{n_i}(D_i) \end{aligned}$$

DONE!

H.W.

$$\left(\prod_i A_i\right)^{op} \cong \prod_i (A_i^{op})$$

call it D_i (it's another div. algebra)

2. The theorem, with more details:

Thm: Suppose A is a s.s. k-alg. with decomp $A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)}$. Then

(1) $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ where $D_i = \left(\mathrm{End}_A(S_1^{(i)})\right)^{op}$ ⓪. Moreover,

A has exactly r simple modules up to isomorphism ①, which are iso. to the modules $D_i^{n_i}$ (module of col. vectors) ②, and $\dim_k(D_i^{n_i}) = n_i \dim_k(D_i)$ ③.

(2) If k is alg. closed and A is f.d., then $A = M_{n_1}(k) \times \cdots \times M_{n_r}(k)$ ④.

Moreover, A has exactly r simples ⑤; the simples are iso to k^{n_i} $(1 \le i \le r)$ ⑥, and $\dim_k(k^{n_i}) = n_i$ ⑦.

Pf (sketch): (0): Follows from Section 1 today

Next time: Corollaries / applications of the A.W. Thm.

(1), (2): Follows since $D_i^{n_i}$ is the only simple of $M_{n_i}(D_i)$ $\forall i$

and since the simples of a direct product are precisely

the inflations of the simples of the direct factors. (Cor. 3.31.)

(3): linear algebra.

(4) – (8): follows from (0) – (3) by Schur's Lemma "Part (2)".

Fact: The decomp. $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for a s.s. algebra A is

unique up to reordering of the factors. i.e., if there is another decomp.

$A \cong M_{n_1'}(D_1') \times \cdots \times M_{n_s'}(D_s')$, then $r = s$ and there is a permutation $\sigma \in S_r$

s.t. $n_i = n'_{\sigma(i)}$ and $D_i = D'_{\sigma(i)}$ $\forall i$.

easy.