AA2. Lecture 35.

04. 15. 2022.

Last time: · Started proof of the "only if" direction of the AW thm.

$$A \text{ s.s} \Rightarrow A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)} \Rightarrow A \cong \left(\mathrm{End}_A(A)\right)^{op} \quad ① ✓$$

$$\cong \left(\prod_i \mathrm{End}_A\left(\bigoplus_j S_j^{(i)}\right)\right)^{op} \quad ②$$

$$\cong \left(\prod_i M_{n_i}(\tilde{D}_i)\right)^{op} \quad ③$$

$$\cong \prod_i M_{n_i}(D_i) \quad ④$$

Today: finish ②.

Proving ② / From $(\mathrm{End}_A(A))^{op}$ to matrices

Ignore the "op" today: focus on $\mathrm{End}_A(A)$.

① data of a set of maps $U_j \to U_i$ $\forall i, j$.

So far we've proved that

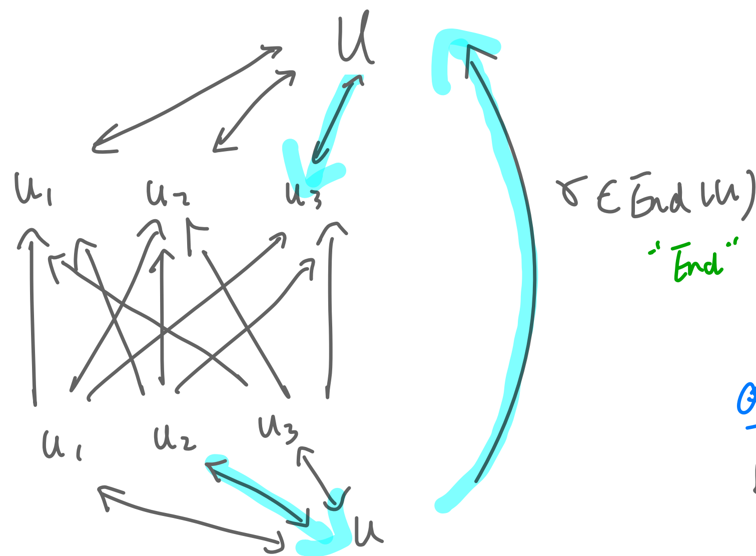
Prop: Let $U_1, U_2, \ldots, U_k$ be A-modules. Then $\Lambda := \left\{ \left[\varphi_{ij} \right]_{i,j} \,\middle|\, \varphi_{ij} \in \mathrm{Hom}(U_j, U_i) \right\}$ is an algebra.

New Prop: $\underbrace{\mathrm{End}_A\left(\bigoplus_{j=1}^{k} U_j\right)}_{\text{"End"}} \cong \underbrace{\Lambda}_{\text{"matrizes"}}$ as algebras. (Point: Data ① is equivalent to Data ②)

② data of a map from $\oplus U_j$ to $\oplus U_j$

A helpful picture: $k=3$. Let $U = \bigoplus_{i=1}^{3} U_i = U_1 \oplus U_2 \oplus U_3$

$U_i \underset{\pi_i}{\overset{\iota_i}{\rightleftarrows}} U$



"matrices"

Λ.

Q1: Given $\gamma \in \mathrm{End}(U)$, how do we get a map from, say, U_2 to U_3?

A: Use $\pi_3 \circ \gamma \circ \iota_2$

Q2: Conversely, if we have homs $\{ f_{ij} : U_j \to U_i : \forall i, j \}$, how can we assemble a map from U to U?

A: Take the "assembled" map:

$$\sum_j \sum_i \iota_i \circ f_{ij} \circ \pi_j$$

Actually, we have just described two inverse isomorphisms between $\mathrm{End}_A(U)$ and Λ.

The actual proof: We will show that the map

$$\Phi : \mathrm{End}_A(\underbrace{\oplus U_i}_{=:U}) \longrightarrow \Lambda = \{ [\varphi_{ij}] : \varphi_{ij} \in \mathrm{Hom}(U_j, U_i) \}$$

$$\gamma \longmapsto [\gamma] := \left[\varphi^{\gamma}_{ij} := \pi_i \circ \gamma \circ \kappa_j \right]_{ij}$$

Where $\kappa_j : U_j \to U$ and $\pi_i : U \to U_i$ are the usual embedding and proj. maps

is an algebra isomorphism. We need to verify the following:

(1). Φ makes sense, i.e., φ^{γ}_{ij} is indeed in $\mathrm{Hom}(U_j, U_i)$ $\forall i, j$ $\forall \gamma \in \mathrm{End}(U)$

domain. ✓ codomain. ✓ hom? ✓.

(2). Φ is linear: $\forall \beta, \gamma \in \mathrm{End}_A(U)$, $\forall a, b \in k$.

$$\varphi^{a\beta + b\gamma}_{ij} = \underset{\substack{\uparrow \\ \text{routine exercise}}}{\cdots} = a\varphi^{\beta}_{ij} + b\varphi^{\gamma}_{ij} \quad \forall i,j. \text{ so } [a\beta + b\gamma] = a[\beta] + b[\gamma].$$

Before moving on, note that $\underbrace{\pi_i \kappa_i = \mathrm{Id}_{U_i} \ \forall i}_{(1)}$, $\underbrace{\sum_i \kappa_i \pi_i = \mathrm{Id}_U}_{(2)}$, and $\underbrace{\pi_i \kappa_j = \delta_{ij} \mathrm{Id}_{U_j}}_{(3)}$ $\forall i,j$.

(3) Φ is unital:

$$\Phi(\mathrm{Id}_U) = \left[\varphi_{ij}^{\mathrm{Id}_V}\right] \text{ where } \varphi_{ij}^{\mathrm{Id}_V} = \pi_i \circ \underbrace{\mathrm{Id}_V \circ \kappa_j}_{\kappa_j} \overset{(3)}{=} \delta_{ij} \mathrm{Id}_{U_j} \qquad \forall i,j.$$

It follows that $\Phi(\mathrm{Id}_U)$ is the identity elt in Λ.

(4) Φ is multiplicative: $\forall \beta, \gamma \in \mathrm{End}(U)$, we have $\Phi(\beta)\Phi(\gamma) = \Phi(\beta\gamma)$ since

$$([\beta][\gamma])_{ij} = \sum_k [\beta]_{ik} [\gamma]_{kj} = \sum_k \pi_i \circ \beta \circ \kappa_k \circ \pi_k \circ \gamma \circ \kappa_j$$

$$\overset{(2)}{=} \pi_i \circ \underbrace{\beta \circ \mathrm{Id}_U \circ \gamma}_{\beta\gamma} \circ \kappa_j = \pi_i \circ (\beta\gamma) \circ \kappa_j = [\beta\gamma]_{ij} \qquad \forall i,j$$

By (1)–(4), it follows that Φ is an alg. hom.

It remains to Φ is bijective.

(5) Surj: Let $[\varphi_{ij}]_{ij} \in \Lambda$. We hope to show that $[\varphi_{ij}]_{ij} = \Phi(\gamma)$ for some $\gamma \in \mathrm{End}(U)$. Consider $\gamma = \underbrace{\sum_{p=1}^{k}\sum_{q=1}^{k} \kappa_q \circ \varphi_{qp} \circ \pi_p}_{\text{from "Q2"}}$. We will show that

$$[\varphi_{ij}] = \Phi(\gamma) = [\gamma]$$

$$[\gamma]_{ij} = \pi_i \circ \left(\sum_p \sum_q \kappa_q \circ \varphi_{pq} \circ \pi_q \right) \circ \kappa_j$$

$$= \sum_{p,q} \pi_i \circ \kappa_p \circ \varphi_{pq} \circ \pi_q \circ \kappa_j$$

$$\overset{③}{=} \delta_{ip}\,\delta_{jq}\, \mathrm{Id}_i \circ \varphi_{ij} \circ \mathrm{Id}_j = \varphi_{ij} \quad \forall i,j.$$

Ex. (see [EH, P108]) $\square$

Note:

- Another way to prove Φ is bij. is to prove that the "assembly map" from Q2 is its two-sided inverse.
- We'll go from Λ to direct products of mat. algebras and finish the proofs next time!