AA2. Lecture 34.

04. 13. 2022.

Last time:

- statement of the A.W. theorem. "A s.s. $\Leftrightarrow \prod_{i=1}^{r} M_{n_i}(D_i)$"
- proof of the $\Leftarrow$ direction.
- Outline of the $\Rightarrow$ direction.

A. s.s.

$$A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)} \Rightarrow$$

$$\begin{aligned}
A &\cong \left(\mathrm{End}_A(A)\right)^{op} && \text{① easy} \\
&\cong \left(\prod_i \mathrm{End}_A\left(\bigoplus_j S_j^{(i)}\right)\right)^{op} && \text{② hard} \\
&\cong \left(\prod_i M_{n_i}(\tilde{D}_i)\right)^{op} && \text{③ hard} \\
&\cong \prod_i M_{n_i}(D_i) && \text{④ easy}
\end{aligned}$$

Today: proof of ①, start ② and ③

1. $A \cong \mathrm{End}_A(A)^{op}$

Prop: For any k-algebra A, we have an isomorphism of algebra

$$\psi : A \to \underbrace{(\mathrm{End}_A(A))^{op}}, \quad a \mapsto \Big(\text{the endo. } r_a : A \to A \text{ given by } r_a(x) = xa \ \forall x \in A\Big)$$

equals $\mathrm{End}_A(A) = \{A\text{-mod homs } f : A \to A\}$, with mult $r \cdot s := s \circ r$

$\downarrow$ composition, the original mult. in $\mathrm{End}_A(A)$.

Straightforward, but instructive. $\downarrow$ You should be able to reproduce this pf!

Pf: We (need to) note the following.

(1) $\forall a \in A$, the assigned map $\psi(a) = r_a$ (right mult. by a) is indeed in the codomain set $(\mathrm{End}_A(A))^{op} = \mathrm{End}_A(A)$, i.e., that r_a is indeed an A-module hom. This is true since (a) r_a is linear since mult in A is bilinear:

$$r_a(cx+dy) = (cx+dy)a = c\cdot(xa) + d\cdot(ya) = c\,r_a(x) + d\,r_a(y) \quad \forall c,d \in k,\ x,y \in A$$

and ab) r_a is an A-mod hom by associativity of mult in A:

$$r_a(b\cdot x) = (b\cdot x)a \xrightarrow{\quad} b\cdot(xa) = b\cdot r_a(x) \qquad \forall b\in A, x\in A$$

(2) The map ψ is a linear map. ie, that $\psi(\lambda a+\mu b) = \lambda\psi(a)+\mu\psi(b)$.

This is an equality of two maps from A to A, so it suffices to check that

$$\begin{aligned}(\psi(\lambda a+\mu b))(x) &= r_{\lambda a+\mu b}(x)\\ &= x(\lambda a+\mu b)\\ &= \lambda xa+\mu xb\\ &= \lambda r_a(x)+\mu r_b(x)\\ &= \lambda(\psi_a(x))+\mu(\psi(b)(x))\\ &= [\lambda\psi(a)](x)+[\mu\psi(b)](x)\\ &= [\lambda\psi(a)+\mu\psi(b)](x) \quad \forall x\in A.\end{aligned}$$

(3) ψ respects multiplication, i.e., $\psi(ab) = \psi(a)\psi(b) \quad \forall\, a, b \in A$.

To do so, note that $\big[\underbrace{\psi(a)\psi(b)}_{\in\, \mathrm{End}_A(A)^{op}}\big](x) = \big[\psi(b) \circ \psi(a)\big](x)$

$$= \psi(b)\big(\psi(a)(x)\big)$$

$$= \psi(b)(xa)$$

$$= (xa)b$$

$$= x(ab)$$

$$= \psi(ab)(x) \quad \forall x \in A$$

(4) ψ preserves unit: $\big(\psi(1_A)\big)(x) = x \cdot 1 = x = id_A(x)$, so $\psi(1_A) = id_A = 1_{\mathrm{End}_A(A)^{op}}$.

By (1), (2), (3) and (4), we conclude that ψ is an alg. hom.

(5). ψ is inj: Suppose $\psi(a) = \psi(b)$ for some $a, b \in A$. Then

$\big(\psi(a)\big)(x) = \big(\psi(b)\big)(x)$ $\forall x \in A$. i.e., $xa = xb$ $\forall x \in A$.

In particular, if we set x to be 1_A we get $1_A a = 1_A b$,

so $a = b$. It follows that ψ is inj.

(6) ψ is surj: We want to show that every element f in the codomain $\mathrm{End}_A(A)^{op}$ equals $\psi(a) = r_a = (\text{right mult by } a)$ for some $a \in A$. So let

$(f : A \to A) \in \mathrm{End}_A(A)^{op}$. Consider $a := f(1_A)$. We hope that $f(x) = xa$ $\forall x \in A$.

the behavior of f is entirely determined by its behavior on 1_A.

This is true because

f is an A-mod. hom

$$f(x) = f(x \cdot 1_A) = x \cdot f(1_A) = x \cdot a = xa = \big(\psi(a)\big)(x).$$ (where the action is on the regular module A.)

By (1)–(6), ψ is a bij. alg. hom, so it's an alg. iso.

2. From $\mathrm{End}_A(A)^{op}$ to matrices

Prop: Let A be an k-algebra. Let $U_1, U_2, \dots, U_k$ be A-modules.

The set
$$\Lambda := \left\{ \begin{bmatrix} & & \\ & \varphi_{ij} & \\ & & \end{bmatrix}_{ij} : \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i) \ \forall i,j \right\}$$
is a k-algebra under the usual mat. addition and multi. formula, where product of matrix entries are given by composition.

E.g. $k = 3$.

$$\begin{bmatrix} f_{11} & f_{12} & \overset{\text{map } U_3 \to U_1}{f_{13}} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \cdot \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} \overset{\text{map } U_1 \to U_1}{f_{11}g_{11} + f_{12}g_{21} + f_{13}g_{31}} & & \\ & & \underset{\mathrm{Hom}(U_3, U_2)}{f_{21}g_{13} + f_{22}g_{23} + f_{23}g_{33}} \\ & & \end{bmatrix}$$

Pf: (sketch)

- Λ is a vector space (under the coordinatewise $+$ and scaling): routine.

- Λ is an algebra: We've argued that Λ is closed under mult.

The alg. axioms are routine to verify, with

$$1_\Lambda = \begin{bmatrix} id_{U_1} & & 0 \\ & \ddots & \\ 0 & & id_{U_k} \end{bmatrix}. \qquad \square$$

Next time:
from Λ to $\left(\prod_i M_i(\widetilde{D_i})\right)^{op}$

Note: If the modules $U_1, \dots, U_k$ are pairwise nonisomorphic simples,

then $\mathrm{Hom}_A(U_j, U_i) = \underbrace{\delta_{ij}}_{\text{Kronecker delta}} \underbrace{\mathrm{End}_A(U_i)}_{}$ $\rightarrow$ some div. alg, e.g. $M_n(k)$, by Schur's Lemma