# AA2. Lecture 33.

04.11.2022.

## Last time:

- HW discussion
- S.s. of path algebras:
  If $Q$ is an acyclic quiver, then $kQ$ is s.s iff $Q_1 = \phi$, in which case $kQ = \prod_{i \in Q_0} k$.

  **Also recall:**
  If $k[x] \ni f = f_1^{a_1} f_2^{a_2} \cdots f_r^{a_r}$ (irr. decomp), then $k[x]/\langle f \rangle$ is s.s $\Leftrightarrow$ $a_i = 1$ $\forall i$.

What about our other main example $kG$, $G$ a gp.

if $G$ is finite

$kG$ is s.s. $\Leftrightarrow$ $\mathrm{char}(k) \nmid |G|$.

The AW. Thm will give an alt. form for $kG$ if $kG$ is s.s.

## Today:

The Artin-Wedderburn Thm

- The statement
- The proof outline

# 1. Statement of the Artin-Wedderburn (AW.) Thm

**Thm:** Let $k$ be a field and $A$ a $k$-algebra

(1) (general case) The algebra $A$ is s.s. iff it is isomorphic to an algebra of the form

$$M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r) \qquad (*)$$

where $n_1, n_2, \cdots, n_r \in \mathbb{Z}_{\geq 1}$ and $D_1, D_2, \cdots, D_r$ are division algebras over $k$. Moreover, if $A$ is s.s then the above decomp is unique up to reordering of the (direct) factors, with $r$ being the number of iso classes of simple A-modules $S_1, S_2, \cdots, S_r$, $n_i$ arising from the dim. of the $i$th simple $S_i$, and $D_i$ arising from the endomorphism ring of the $i$th simple $S_i$ $\forall i$.

(2) (Special case) If $k$ is algebraically closed, then $A$ is s.s.

iff $A$ is isomorphic to an algebra of the form

$$M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k)$$

where $n_1, n_2, \ldots, n_r \in \mathbb{Z}$. The "moreover" part of (1) still holds.

## 2. Proof outline of the A.W. thm.

We prove the general case.

(a) The "if" direction: "$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ is s.s".

(easier)

**Pf:** Since direct product of s.s. algebras are s.s., it suffices to show that each comp $M_{n_i}(D_i)$ is s.s.

We can prove $M_n(D)$ is s.s for any $n \in \mathbb{Z}_{\geq 1}$ and division alg. $D/k$ in the same way as we proved $M_n(k)$ is s.s:

We show $\underbrace{M_n(D)}_{\text{reg. module}} \cong \bigoplus_{i=1}^{n} C_i$ where $C_i = \left\{ \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c|c} 0 & 0 & \cdots & * & 0 & \cdots & 0 \\ & & & * & & & \\ & & & * & & & \end{array}\right] \\ \text{ith col} \end{array} \in M_n(D) \right\}$

need to check: · each $C_i$ is a submodule. easy

· the internal direct sum decomp: easy, lin algebra

- each $C_i$ is simple: use the strong cyclicity test.

Take $0 \neq v \in C_i$ Say $v = \left[\begin{array}{c|c|c} 0 & \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix} & 0 \end{array}\right]$ with $a_j \neq 0$ for some $j$.

Then $E_{1j} \cdot v = \left[\begin{array}{c|c|c} 0 & a_j e_1 & 0 \end{array}\right]$

and hence $(a_j^{-1} E_{1j}) \cdot v = \left[\begin{array}{c|c|c} 0 & e_1 & 0 \end{array}\right]$, where $a_j^{-1}$ exists since $a_j$ is a nonzero elt in the division algebra.

But then

$$E_{i1}\left(\left[\begin{array}{c|c|c} 0 & e_1 & 0 \end{array}\right]\right) = \left[\begin{array}{c|c|c} 0 & e_i & 0 \end{array}\right] \quad \forall\, 1 \leq i \leq n.$$

It follows that $v$ generates the entire $C_i$ in the sense that

$M_n(D) \cdot v = C_i$, so $C_i$ is simple.

We are done with the "if" part of the A.W. Thm.

(b) The "only if" part: given a s.s. algebra over $k$, why is it iso to an algebra of the form $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$?

Sketch of the proof outline: Suppose $A$ is s.s. Then the regular module $A$ decomposes into simples, say as

$$A = \left(S_1^{(1)} \oplus S_2^{(1)} \oplus \cdots \oplus S_{n_1}^{(1)}\right) \oplus \left(S_1^{(2)} \oplus S_2^{(2)} \oplus \cdots \oplus S_{n_2}^{(2)}\right) \oplus \cdots \oplus \left(S_1^{(r)} \oplus S_2^{(r)} \oplus \cdots \oplus S_{n_r}^{(r)}\right).$$

Note that the multiplicities $n_1, \cdots, n_r$ are uniquely determined because $n_j$ is just the multiplicity of the comp. factor $S^{(j)}$ in a comp. series of $A$ $\forall j$.

$$A = \left(S_1^{(1)} \oplus S_2^{(1)} \oplus \cdots \oplus S_{n_1}^{(1)}\right) \oplus \left(S_1^{(2)} \oplus S_2^{(2)} \oplus \cdots \oplus S_{n_2}^{(2)}\right) \oplus \cdots \oplus \left(S_1^{(r)} \oplus S_2^{(r)} \oplus \cdots \oplus S_{n_r}^{(r)}\right)$$

Using the above decomposition, we will show

pf soon

$$A \overset{\text{pf soon}}{\cong} \left(\underbrace{\mathrm{End}_A(A)}\right)^{op}$$

op of an algebra: same algebra except "new ab = old ba".

→ general fact about algebras

→ the set of end. of the regular module $V = A$. i.e. A-mod homs $f: A \to A$

$$= \left(\mathrm{End}_A\left(\bigoplus_{i=1}^{r}\left(\bigoplus_{j=1}^{n_i} S_j^{(i)}\right)\right)\right)^{op}$$

→ rewriting

hard

$$\cong \left(\prod_{i=1}^{r} \mathrm{End}_A\left(\bigoplus_{j=1}^{n_i} S_j^{(i)}\right)\right)^{op}$$

→ Schur's Lemma (non iso simples have no homs in between), roughly

hard

$$\cong \left(\prod_{i=1}^{r} M_{n_i}(\underbrace{\tilde{D}_i}_{})\right)^{op}$$

come from Schur's Lemma

→ Schur's Lemma + generalization of the fact that $\mathrm{End}_k(V) \cong M_n(k)$ for a $k$-vec space $V$ w/ dim. $n$.

continued: $\cong \left( \prod_{i=1}^{r} M_{n_i}(\widetilde{D}_i) \right)^{op}$

easier

$\cong \prod_{i=1}^{r} \underline{\left( M_{n_i}(\widetilde{D}_i) \right)^{op}}$ easier $\cong \prod_{i=1}^{r} \underline{M_{n_i}(D_i)}$ $\to$ desired form. Done.

$\to$ cleaning up the op and div. algebras

no op, $D_i$ coming from $\widetilde{D}_i$.

Next time(s): proofs of the ingredients/isos.