AA2. Lecture 32. Final Exam: take-home. cumulative 04. 08. 2022.

available 5pm Apr 30 → 5pm May 1 on Canvas

Last time: · Jacobson radicals and semisimplicity

Summary of properties of Jacobson radicals of a finite-length algebra A.

(a) J is an intersection of finitely many max. left ideals. ($J = J(A)$)

(b) J kills all simples, and $J = \bigcap_{S \text{ simple}} \mathrm{Ann}_A(S)$.

(c) – (d), (d') J is the largest nilpotent two-sided ideal.

(e) – (f): J is the smallest two-sided ideal w/ a s.s. quotient.

(g) – (h): s.s. criteria: A is s.s $\Leftrightarrow J = 0$, an A-module V is s.s. $\Leftrightarrow J \cdot V = 0$.

· s.s. of $A = k[x]/\langle f \rangle$ via Jacobson radicals:

if $f = \underbrace{f_1}_{\text{irr.}}^{a_1} \cdots \underbrace{f_r}_{\text{irr.}}^{a_r}$ in the unique decomp. into irr., then A is s.s. iff $a_i = 1 \ \forall i$.

Today: · HW. · Jacobson radical and s.s. of path algebras.

1. Homework

P4: Suppose A is of finite length. Show that $J = J(A)$ contains every left nilpotent left ideal I of A.

(d') of our theorem.

$\overset{\text{def}}{=} \bigcap_{\substack{\text{all} \\ \text{max. left ideals } M \text{ of } A.}}$

Strategy: Take $x \in I$. We want to show $x \in J$.

Suppose not. i.e., Suppose $x \notin J$. then $x \notin M$ for some maximal left ideal M of A.

This implies that $M + Ax \supsetneq M$, so $M + Ax = A$. In particular,

$\exists\, m \in M,\ r \in A$ s.t. $m + rx = 1$ $\quad (m = 1 - rx)$.

Trick: Since $x \in I$, $rx \in I$. Since I is nilp., $(\underbrace{rx}_{I})^k = 0$ for some k large enough,

So

$$1 = 1 - (rx)^k \overset{\text{algebra}}{=} \left(1 + (rx) + (rx)^2 + \cdots + (rx)^{k-1}\right)\underbrace{(1 - rx)}_{m}$$

$\rightarrow$ use this to argue that $Am = A$ and hence $M = A$.

P5. A : finite-length commutative algebra,

$x \in A$: nilp. elt, i.e., $x^k = 0 \ \forall k >> 0$.

Want: $x \in J$.

Strategy: we just showed that every nilp. ideal I is contained in J, so it suffices to show that x is in some nilp. ideal I.

obvious guess: $I_x := \langle x \rangle$ works, i.e., is I_x nilp?

Answer: yes, use commutativity; every elt in I_x^k is a linear comb of things of the form $z := z_1 \cdots z_k$ where $z_i = a_i x \ \forall i$.

Now, $z = (a_1 x)(a_2 x) \cdots (a_k x)$. $\rightarrow$ use commutativity to show $z = 0$.

2. Jacobson radical and semisimplicity of path algebras

Prop: Let Q be an acyclic quiver and let $A = kQ$. Then the Jacobson radical

$J = J(A)$ is the subspace of A spanned by all paths of positive length in Q.

Rmk: For any algebra A, isomorphic A-modules M, N have equal annihilators by the principle of preservation of scalar actions:

$$\mathrm{Ann}_A(M) = \{a \in A : a \cdot M = 0\} = \{a \in A : a \text{ acts as } 0 \text{ on } M\} = \{a \in A : a \cdot N = 0\} = \mathrm{Ann}_A(N).$$

Pf: Recall (Lecture 20, Mar. 02) that the modules $S_i = Ae_i / J_i$ where $J_i = (Ae_i)_{\geq 1}$ $(i \in Q_0)$ form a complete set of simple modules of A up to isomorphism. Thus, we have

$$J = \bigcap_{i \in Q_0} \mathrm{Ann}(S_i) \text{ by the earlier remark.}$$

(What is $\mathrm{Ann}_A(S_i)$?) We note that

span of all paths except e_i.

$$\mathrm{Ann}_A(S_i) = \underbrace{\mathrm{Ann}_A\left(Ae_i/J_i\right)}_{X} = \underbrace{J_i \oplus \left(\bigoplus_{j\neq i} Ae_j\right)}_{Y}.$$

Ex:
- prove $Y \subseteq X$ using def of the action $A \circlearrowright Ae_i/J_i$
- deduce $Y = X$ via a dimension argument.

It follows that

$$J = \bigcap_{i\in Q_0} \mathrm{Ann}_A(S_i) = \text{the span of all positive-length paths.}$$

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

$i = 1$: $\mathrm{Span}\{e_2, e_3, \alpha, \beta, \alpha\beta\}$

$i = 2$: $\mathrm{Span}\{e_1, e_3, \alpha, \beta, \alpha\beta\}$

$i = 3$: $\mathrm{Span}\{e_1, e_2, \alpha, \beta, \alpha\beta\}$

Corollary: (Corollary 4.27) In the setting of the prop. (Q acyclic)

We have (1) $A = kQ$ is s.s iff Q has no arrows.

(2) Moreover, if Q has no arrows, then $kQ \overset{\wedge}{=} \underbrace{k \times k \times \cdots \times k}_{|Q_0| \text{ copies}}$.

$\updownarrow$

$\circ \quad \circ \quad \cdots \quad \circ$

Pf: (1) A is s.s $\iff J(A) = 0$

$\iff$ (Span of all positive length paths) $= 0$

$\iff$ Q has no positive path

$\iff$ Q has no arrows.

(2) Ex: find an iso, with each copy of k corresponding to a vertex.

Next time: Ch5 — the A.W. theorem.