

available 5pm Apr 30 → 5pm May 1 on Canvas

Last time: · Jacobson radicals and semisimplicitySummary of properties of Jacobson radicals of a finite-length algebra A .(a) J is an intersection of finitely many max. left ideals, $J = J(A)$ (b) J kills all simples, and $J = \bigcap_{S \text{ simple}} \text{Ann}_A(S)$.(c) - (d), (d') J is the largest nilpotent two-sided ideal.(e) - (f) : J is the smallest two-sided ideal w/ a s.s. quotient.(g) - (h) : s.s. criteria : A is s.s. $\Leftrightarrow J=0$, an A -module V is s.s.· s.s. of $A = k[x]/\langle f \rangle$ via Jacobson radicals: $\bigcap_{J} V = 0$.if $f = \underbrace{f_1^{a_1}}_{\text{irr.}} \cdots \underbrace{f_r^{a_r}}_{\text{irr.}}$ in the unique decomp. into irr., then A is s.s. iff $a_i = 1 \forall i$.Today: · HW. · Jacobson radical and s.s. of path algebras.

1. Homework

P4: Suppose A is of finite length. Show that $J = J(A)$ contains every left nilpotent left ideal I of A . (d') of our theorem.

Strategy: Take $x \in I$. We want to show $x \in J$. $\stackrel{\text{def}}{=} \bigcap$ all max. left ideals M of A .

Suppose not. i.e., suppose $x \notin J$. then $x \notin M$ for some maximal left ideal M of A .

This implies that $M + Ax \neq M$, so $M + Ax = A$. In particular,

$$\exists m \in M, r \in A \text{ s.t. } m + rx = 1 \quad (m = 1 - rx).$$

Trick: Since $x \in I$, $rx \in I$. Since I is nilp., $(rx)^k = 0$ for some k large enough,

so

$$1 = 1 - (rx)^k \stackrel{\text{algebra}}{=} \left(1 + (rx) + (rx)^2 + \dots + (rx)^{k-1} \right) \underbrace{(1 - rx)}_m \rightarrow \text{use this to argue that } Am = A$$

and hence $M = A$.

P5. A : finite-length, commutative algebra,

$x \in A$: nilp. elt, i.e., $x^k = 0 \quad \forall k \gg 0$.

Want: $x \in J$.

Strategy: we just showed that every nilp. ideal I is contained in J ,
so it suffices to show that x is in some nilp. ideal I .

obvious guess: $I_x := \langle x \rangle$ works, i.e., is I_x nilp.?

Answer: yes, use commutativity: every elt in I_x^k is a linear comb of
things of the form $z := z_1 \cdots z_k$ where $z_i = a_i x \quad \forall i$.

Now, $z = (a_1 x)(a_2 x) \cdots (a_k x) \rightarrow$ use commutativity to show
 $z = 0$.

2. Jacobson radical and semisimplicity of path algebras

Prop. Let \mathcal{Q} be an acyclic quiver and let $A = k\mathcal{Q}$. Then the Jacobson radical

$J = J(A)$ is the subspace of A spanned by all paths of positive length in \mathcal{Q} .

Remk. For any algebra A , isomorphic A -modules M, N have equal annihilators by the principle of preservation of scalar actions:

$$\text{Ann}_A(M) = \{a \in A : a \cdot M = 0\} = \{a \in A : a \text{ acts as } 0 \text{ on } M\} = \{a \in A : a \cdot N = 0\} = \text{Ann}_A(N).$$

Pf. Recall (Lecture 20, Mar. 02) that the modules $S_i = \frac{Ae_i}{J_i}$ where $J_i = (Ae_i)^{\perp}$ ($i \in \mathcal{Q}_0$)

form a complete set of simple modules of A up to isomorphism. Thus, we have

$$J = \bigcap_{i \in \mathcal{Q}_0} \text{Ann}(S_i) \text{ by the earlier remark.}$$

(What is $\text{Ann}_A(S_i)$?) We note that

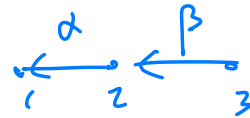
span of all paths except e_i .

$$\text{Ann}_A(S_i) = \underbrace{\text{Ann}_A(Ae_i/J_i)}_X = J_i \oplus \underbrace{\left(\bigoplus_{j \neq i} Ae_j \right)}_Y$$

Ex: · prove $Y \subseteq X$ using def of the action $A \curvearrowright Ae_i/J_i$
 · deduce $Y = X$ via a dimension argument.

It follows that

$J = \bigcap_{i \in \mathbb{Q}_0} \text{Ann}_A(S_i) =$ the span of all positive-length paths.



$i = 1$: $\text{Span} \{ e_2, e_3, \alpha, \beta, \alpha\beta \}$

$i = 2$: $\text{Span} \{ e_1, e_3, \alpha, \beta, \alpha\beta \}$

$i = 3$: $\text{Span} \{ e_1, e_2, \alpha, \beta, \alpha\beta \}$

Corollary: (Corollary 4.27) In the setting of the prop. (Q acyclic)

We have (1) $A = kQ$ is s.s. iff Q has no arrows.

(2) Moreover, if Q has no arrows, then $kQ \cong k \times k \times \dots \times k$.

Pf. (1) A is s.s. $\Leftrightarrow J(A) = 0$

\Leftrightarrow (Span of all positive length paths) = 0

$\Leftrightarrow Q$ has no positive path

$\Leftrightarrow Q$ has no arrows.

(2) Ex. find an iso, with each copy of k corresponding to a vertex.

Next time: Ch5 — the A.W. theorem.