AA2. Lecture 31. 04.06.2022.

Last time:

- proved the following properties of the Jacobson radical $J = J(A)$ of a finite-length algebra A:

(a) J equals an intersection of finitely many max. left ideals.

(b) $J = \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$.

(c) J is a two-sided ideal.

(d) J is nilpotent, and $J^n = 0$ where $n = \mathrm{length}(A)$.

Also, fact: J contains every nilp. left ideal of A.

Today:

- proofs of more properties of J (they all concern s.s.)
- an application of J: s.s. of $k[x]/\langle f \rangle$, $f \in k[x]$.

1. More properties of $J(A)$ → $J(A)$ and semisimplicity

We may assume that r is minimal, s.t. that $\bigcap_{j \neq i} M_j \not\subseteq M_i$ $\forall i$.

Let A be an algebra of finite length. Let $J = J(A)$.

(e). A/J is s.s.

Pf: By (a), $J = \bigcap_{i=1}^{r} M_i$ for some max. left ideals $M_1, \dots, M_r$ of A.

We will show that there is A-module isomorphism

$$\Phi : A/J \longrightarrow A/M_1 \oplus A/M_2 \oplus \cdots \oplus A/M_r$$

given by $\quad a+J \longmapsto (a+M_1, a+M_2, \dots, a+M_r) \quad \forall a \in A$.

It would follow, since each A/M_i is a simple A-module, that A/J is a s.s. A-module. Since $J \cdot (A/J) \subseteq J \cdot A/J \subseteq J/J = 0$, it follows that A/J is a s.s. A/J-modules.

We prove Φ is an A-module iso. as follows: $\boxed{a+J \longmapsto (a+M_1, a+M_2, \dots, a+M_r)}$

(1) Φ is well-defined:

$$a+J = b+J \Rightarrow a-b \in \underset{\bigcap M_i}{\overset{\parallel}{J}} \Rightarrow a-b \in M_i \ \forall i \Rightarrow a+M_i = b+M_i \ \forall i \Rightarrow \Phi(a) = \Phi(b). \checkmark$$

(2) Φ is an A-module hom:

E.x. Routine, need $\Phi(r\cdot(a+J)) = r\cdot\Phi(a+J)$. $\leftarrow$ follows since Φ is "natural".

(3) Φ is inj: The kernel is trivial since

$$a+J \in \ker\Phi \Rightarrow a+M_i = 0 \ \forall i \Rightarrow a \in M_i \ \forall i \Rightarrow a \in \bigcap_{i=1}^{r} M_i = J \Rightarrow a+J = 0.$$

(4) Φ is surj: It suffices to show that $(0, 0, \dots, 0, \underset{\substack{\downarrow \\ i\text{th position}}}{1+M_i}, 0, \dots, 0) \in \operatorname{Im}\Phi$

for every i. To this end, consider the module $\bigcap_{j\neq i} M_j$.

By assumption, $\bigcap_{j \neq i} M_j \not\subseteq M_i$. Thus, since M_i is a maximal ideal, we have

$$M_i + \bigcap_{j \neq i} M_j = A.$$

In particular, we have $1 = m_i + y$ for some $m_i \in M_i$, $y \in \bigcap_{j \neq i} M_j$.

But then

$$\begin{aligned} \Phi(y + J) &= (y + M_1, y + M_2, \cdots, y + M_i, \cdots, y + M_r) \\ &= (0, 0, \cdots, 1 - m_i + M_i, \cdots, 0) \\ &= (0, 0, \cdots, 1 + M_i, \cdots, 0), \end{aligned}$$

as desired.

We are done. $\square$

(f) If A/I is a s.s algebra for a two-sided ideal I, then $J \subseteq I$.

Pf: Suppose A/I is s.s. Then $A/I = S_1 \oplus \cdots \oplus S_k$ as A/I-modules where $S_1, \ldots, S_k$ are simple A/I-modules.

Note that (Lemma 3.5 ?, inflation) S_i is also a simple A-module $\forall i$, so by (b) we have $J \cdot S_i = 0$ $\forall i$. Thus, we have

$$J \cdot (A/I) = J \cdot (S_1 \oplus \cdots \oplus S_k) = 0,$$

i.e., we have $J = J \cdot A \subseteq I$.

Again: (e) and (f) say that J is the minimum ideal w/ a s.s. quotient.

(g) A is s.s. iff $J = 0$.

pf: If A is s.s., then $A/0 \cong A$ is s.s., so by (f) we have $J \subseteq 0$, so $J = 0$.

Conversely, if $J = 0$, then $A \cong A/0 = A/J$, which is s.s. by (e).

(h) An A-module V is s.s iff $J \cdot V = 0$.

pf: ($\Rightarrow$) If V is s.s. then $V = \bigoplus_{i=1}^{r} S_i$, S_i a simple A-module $\forall i$.

But $J \cdot S_i = 0$ $\forall i$ by (b), therefore $J \cdot V = 0$.

($\Leftarrow$) If $J \cdot V = 0$, then V is an A/J-module. Since A/J is s.s by (e), it follows that V is a s.s. A/J-module. It further follows that that V is a s.s. A-module (killed by J). $\square$

2. An application: semisimplicity of $k[x]/\langle f\rangle$

Next time: $J(A)$ and s.s of path algebras $A = kQ$.

Prop: Let $f \in k[x]$, and let $f = f_1^{a_1} f_2^{a_2} \cdots f_m^{a_m}$ where $f_1, \dots, f_m$ are pairwise coprime irreducible polynomials. Then the algebra $k[x]/\langle f\rangle$ is s.s iff $a_1 = a_2 = \cdots = a_m = 1$.

Pf: Recall that the maximal ideals of $k[x]/\langle f\rangle$ are precisely the ideals of the form $\langle h\rangle/\langle f\rangle$ where h is an irreducible factor of f, i.e., they are exactly $\langle f_1\rangle/\langle f\rangle, \langle f_2\rangle/\langle f\rangle, \dots, \langle f_r\rangle/\langle f\rangle$. It follows that the Jacobson radical of $k[x]/\langle f\rangle$ is $\bigcap_{i=1}^{r} \langle f_r\rangle/\langle f\rangle = \langle LCM(f_1, \dots, f_r)\rangle/\langle f\rangle = \langle f_1 f_2 \cdots f_r\rangle/\langle f\rangle$.

The last quotient is 0 iff $f = f_1 f_2 \cdots f_r$, i.e., iff $a_i = 1$ $\forall i$, so $k[x]/\langle f\rangle$ is s.s. iff $a_i = 1$ $\forall i$ by Part (g) of the last thm. $\square$